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think!
NEW SYLLABUS MATHEMATICS

8th Edition

For
Cambridge O Level and Cambridge IGCSE Mathematics

1

Dr Yeap Ban Har • Dr Joseph B. W. Yeo • Dr Choy Ban Heng
Teh Keng Seng • Wong Lai Fong • Sharon Lee • Ong Chan Hong

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PREFACE

think! Mathematics is a series of textbooks specially designed to provide students valuable learning experiences by engaging their minds and hearts as they learn mathematics.

The features of this textbook series reflect the important shifts towards the development of 21st century competencies and a greater appreciation of mathematics, as articulated in the Singapore mathematics curriculum and other international curricula. Every chapter begins with a Chapter Opener and an Introductory Problem to motivate the development of the key concepts in the topic. The Chapter Opener gives a coherent overview of the big ideas that will frame the study of the topic, while the Introductory Problem positions problem solving at the heart of learning mathematics. Two key considerations guide the development of every chapter – seeing mathematics as a tool and as a discipline. Opportunities to engage in Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Tasks are woven throughout the textbook to enhance students' learning experiences. Stories, songs, videos and puzzles serve to arouse interest and pique curiosity. Real-life examples serve to influence students to appreciate the beauty and usefulness of mathematics in their surroundings.

Underpinning the writing of this textbook series is the belief that all students can learn and appreciate mathematics. Worked Examples are carefully selected, questions in the Reflection section prompt students to reflect on their learning, and problems are of varying difficulty levels to ensure a high baseline of mastery, and to stretch students with special interest in mathematics. The use of ICT helps students to visualise and manipulate mathematical objects with ease, hence promoting interactivity.

We hope you will enjoy the subject as we embark on this exciting journey together to develop important mathematical dispositions that will certainly see you through beyond the examinations, to appreciate mathematics as an important tool in life, and as a discipline of the mind.

KEY FEATURES

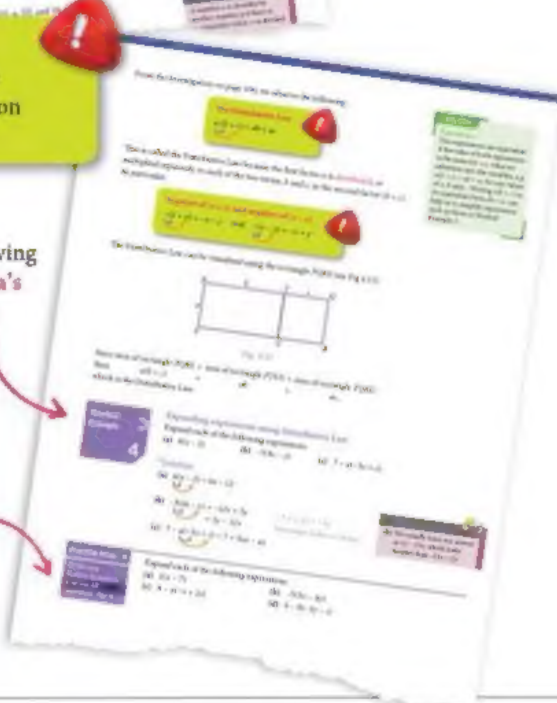
Chapter Opener
gives students an overview of the topic. It includes *rationales* for learning the chapter to arouse students' *interest* and **big ideas** that *connect* the concepts within the chapter or with other chapters.

Learning Outcomes
help students to be aware of what they are about to study so as to *monitor* their progress.

Introductory Problem
provides students with a more specific *motivation* to learn the topic, using a problem that helps develop a concept, or an application problem that students will revisit after they have gained necessary knowledge from the chapter.

Recap
revisits relevant prerequisites at the beginning of the chapter or at appropriate junctures so that students are *ready* to learn new knowledge built on their existing schema.

Important Results
summarise important concepts or formulae obtained from Investigation, Class Discussion or Thinking Time.

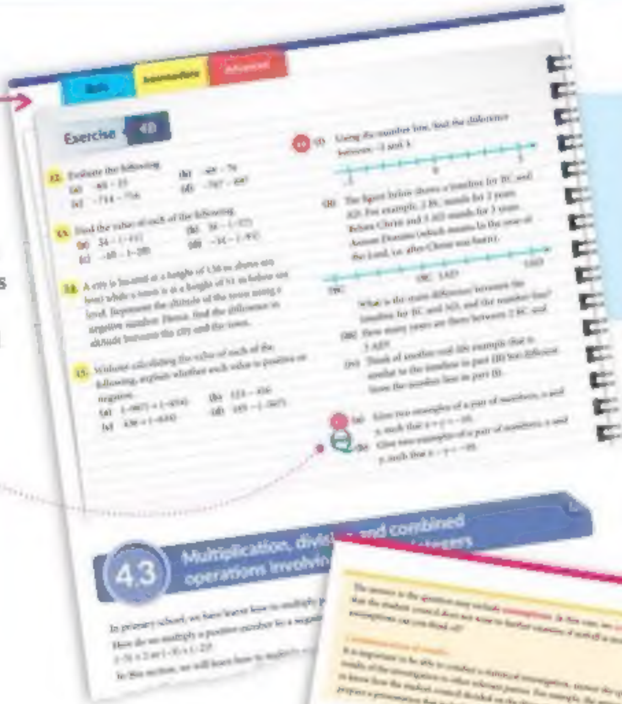


Worked Example
shows students how to present their working clearly when solving related *problems*. In more challenging worked examples, **Pólya's Problem Solving Model** is used to help students learn how to address a problem.

Practise Now
consists of questions that help students achieve *mastery* of procedural *skills*. Puzzles are sometimes used for consolidation to make practice *motivating* and fun.

Similar and Further Questions
follow after Practise Now to help teachers select appropriate questions for students' self-practice.

Exercise
questions are classified into three levels of difficulty – **Basic**, **Intermediate** and **Advanced**. Questions at the Basic level are usually short-answer items to test basic concepts and skills. The Intermediate level contains more structured questions, while the Advanced level involves applications and higher order thinking skills.

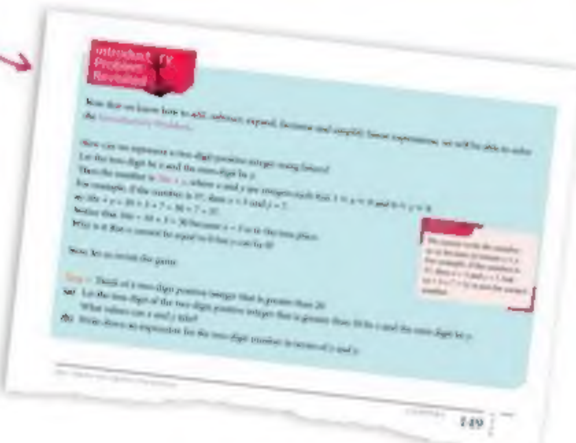


Explanation Questions
require students to communicate their explanations in writing and are spread throughout the textbook.

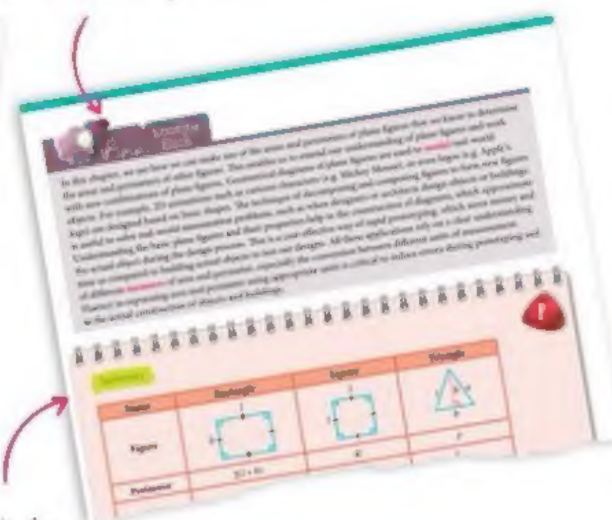
Open-ended Problems
are mathematics problems with more than one correct answer. Solving such problems expose students to real-world problems.

Performance Task
consists of mini-projects designed to develop research and presentation skills of students, through writing a report and/or giving an oral presentation.

Introductory Problem Revisited
revisits an application-based Introductory Problem later in the chapter. This is absent if the Introductory Problem leads directly to the development of a concept.



Looking Back
complements the Chapter Opener and helps students internalise the **big ideas** that they have learnt in the chapter.



Summary
compounds the key concepts taught in the chapter in a succinct manner. Questions are included to help students *reflect* on their learning.

Investigation

Guided investigation provides students the relevant *learning experiences* to explore and discover important mathematical *concepts*. It usually takes the **Concrete-Pictorial-Abstract (C-P-A)** approach to help students construct their knowledge meaningfully. The connections between concrete experiences (manipulative or examples), different pictorial representations and symbolic representations are explicitly made. Some investigations may also involve the use of **Information and Communication Technology (ICT)**.

Class Discussion

Questions are provided to **engage** students in discussion, with the teacher acting as the facilitator. Class discussions provide students the relevant *learning experiences* to think and *reason* mathematically, enhance their oral *communication* skills, and learn new *concepts* and *skills*.

Thinking Time

Key questions are included at appropriate junctures to provide students the relevant *learning experiences* to think critically on their own before sharing their thoughts with their classmates. Mathematical fallacies are sometimes included to check and test students' understanding.

Journal Writing

Journal writing provides opportunities for students to *reflect* on their learning and to *communicate* mathematically in writing. It can also be used as a formative assessment for the teacher to provide feedback for their students.

Reflection

Students are usually required to reflect on what they have learnt at the end of each section so as to *monitor* and *regulate* their own learning. The reflection questions provided can be generic prompts or specific to the topics in the section or chapter, to check if students have understood the key ideas.

MARGINAL NOTES

Big Idea

This provides additional details of the big idea mentioned in the main text.

Recall

Unlike the key feature 'Recap' in the main text, this contains just-in-time recall of prerequisite knowledge that students have already learnt.

Attention

This contains important information that students should know.

Information

This includes information that may be of interest to students.

Reflection

This guides students to think about different methods used to solve a problem.

Problem-solving Tip

This guides students on how to approach a problem in Worked Examples or Practise Now.

Internet Resources

This guides students to search the Internet for valuable information or interesting online games for their independent and self-directed learning.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about mathematics as enrichment for students.

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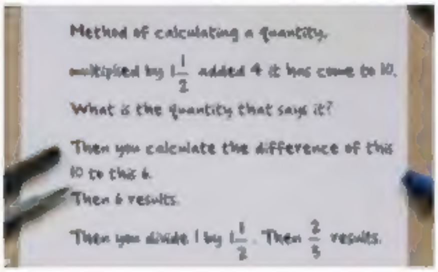

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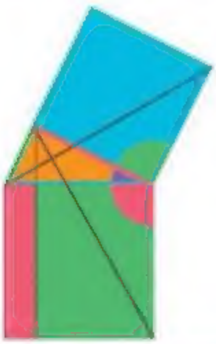

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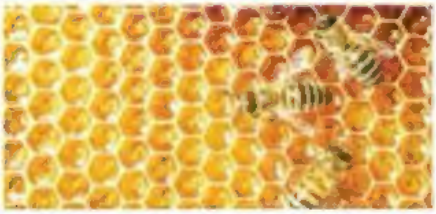

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CHAPTER 1

Primes, Highest Common Factor and Lowest Common Multiple

Sensitive data that are transferred over the Internet, such as credit card numbers and passwords, have to be encrypted.

In 1978, Ronald Rivest, Adi Shamir and Leonard Adleman publicly described the RSA algorithm, which is the basis for public key cryptography. RSA provides a method to ensure the secure encryption of data that even the most advanced computers will take years to crack. It makes use of a complicated theorem involving a type of numbers called prime numbers.

Prime numbers can be said to be the building blocks of all whole numbers greater than 1. Every whole number greater than 1 is either a prime or a unique product of primes! This has various applications in both mathematics and the real world. In this chapter, we are going to explore how whole numbers can be broken down into its building blocks.

Learning Outcomes

What will we learn in this chapter?

- What prime and composite numbers are
- How to find the square root of a perfect square and the cube root of a perfect cube
- Why highest common factor (HCF) and lowest common multiple (LCM) have useful applications in real life

Introductory Problem

Albert has a vanguard sheet with a length of 64 cm and a breadth of 48 cm. He wants to cut it into squares that are as big as possible, without any leftover vanguard sheet.

(i) What is the length of each square?

(ii) How many squares can he cut altogether?

In this chapter, we are going to learn about prime numbers, the highest common factor (HCF) and the lowest common multiple (LCM), which can help us solve these kinds of problems.

1.1 Prime numbers

A. Whole numbers and factors (Recap)

Let us recap on what we know about whole numbers and factors. What are the missing numbers below?

Examples of whole numbers are 0, 1, 2, ☐, 4, ☐, 6, …

Even numbers are whole numbers that are divisible by 2, e.g. 0, 2, 4, ☐, 8, ☐, 12, 14, ….

Odd numbers are whole numbers that are not divisible by 2, e.g. 1, 3, 5, ☐, 9, 11, ☐, 15, 17, ….

Can you find the factors of 18?

$$\begin{aligned} 18 &= 1 \times 18 \\ &= 2 \times \square \\ &= \square \times 6 \end{aligned}$$

Therefore, the factors of 18 are 1, 2, ☐, 6, ☐ and 18.

Is 18 divisible by each of its **factors**?

> **Problem-solving Tip**
>
> A number n is divisible by another number p if there is no remainder when n is divided by p.

B. Classifying whole numbers

Whole numbers can be divided (or classified) into two groups: even numbers and odd numbers.

Another way to classify whole numbers is to group them by the number of factors they have.

Classifying whole numbers

1. Find the factors of the numbers in Table 1.1.

Number	Working	Factors	Number	Working	Factors
1	1 is divisible by 1 only	1	11		
2	$2 = 1 \times 2$	1, 2	12		
3			13		
4	$4 = 1 \times 4 = 2 \times 2$	1, 2, 4	14		
5			15		
6			16		
7			17		
8			18	$18 = 1 \times 18 = 2 \times 9 = 3 \times 6$	1, 2, 3, 6, 9, 18
9			19		
10			20		

Table 1.1

2. Classify the numbers in Table 1.1 into 3 groups.

   Group A contains a number with exactly *1 factor*:

   Group B contains numbers with exactly *2 different factors*:

   Group C contains numbers with *more than 2 different factors*:

3. Is 0 divisible by 1, 2, 3, 4, etc.? How many factors does 0 have?

C. Prime numbers and composite numbers

In the above Investigation, the number in **Group A** does not have a name.

The numbers in **Group B** are known as **prime numbers** (or **primes**).

The numbers in **Group C** are called **composite numbers**.

Composite numbers are *composed* (or made up) of the product of at least two primes, e.g. $6 = 2 \times 3$ and $18 = 2 \times 3 \times 3$.

- A **prime number** is a whole number that has *exactly 2 different factors*, 1 and itself.
- A **composite number** is a whole number that has *more than 2 different factors*.
- 0 and 1 are neither prime nor composite.

1. Explain why 0 and 1 are neither prime nor composite.
2. Yasir says that if a whole number is not prime, then it must be composite.

   Do you agree? Explain your answer.

In Table 1.1 in the Investigation on page 3, we identified prime numbers by finding all the factors of the numb
An easier way to *sieve out* prime numbers less than or equal to 100 is called the **Sieve of Eratosthenes**.

Sieve of Eratosthenes

1. We will sieve out the prime numbers in Fig. 1.1 by circling those which are prime and crossing out those which are not. Follow the instructions below.
   (a) Cross out 1.
   (b) Circle 2.
   Cross out all the other multiples of 2 (because they are not primes: why?).
   (c) The next number that is not crossed out, i.e. 3, is a prime. Circle 3.
   Cross out all the other multiples of 3.
   (d) The next number that is not crossed out, i.e. 5, is a prime. Circle 5.
   Cross out all the other multiples of 5.
   (e) Continue doing this until all the numbers have either been circled or crossed out.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fig. 1.1

2. Answer the following questions.
   (a) What is the smallest prime number?
   (b) What is the largest prime number less than or equal to 100?
   (c) How many prime numbers are less than or equal to 100?
   (d) Is every odd number a prime number? Explain.
   (e) Is every even number a composite number? Explain.
   (f) For a prime number greater than 5, what can the last digit be? Explain.

Just For Fun

(a) **Twin primes** are prime numbers that differ by 2, such as 5 and 7. List two other pairs of twin primes.

(b) **Cousin primes** are prime numbers that differ by 4, such as 7 and 11. List two other pairs of cousin primes.

(c) There are prime numbers that differ by 6, such as 2011 and 2017. List two other such pairs.

Product of prime numbers

Can the product of two prime numbers be

(a) an odd number? (b) an even number? (c) a prime number?

Explain your reasoning or give a counterexample.

D. Trial division

To test whether a number is prime or composite, we have to learn a new concept called **square root**.

For example, $5 \times 5 = 25$. We say that the **square** of 5, or 5 **squared**, is 25.

If we do the reverse, we get $\sqrt{25} = 5$. We say that the *square root* of 25 is 5.

What is the value of $\sqrt{9}$ and of $\sqrt{16}$?

What about $\sqrt{47}$? Its value will not be a whole number. Why?

Use a calculator to evaluate $\sqrt{47}$ by pressing the $\sqrt{}$ key. Did you get 6.9 (to 1 d.p.)?

Attention

6.9 (to 1 d.p.) means the answer 6.9 is correct to one decimal place.

To find out whether a number is prime or composite, we check if it is divisible by all the prime numbers less than or equal to its square root. Why?

This method is called **trial division**.

Worked Example 1 shows how we can test whether a number is prime or composite.

Test for prime number

Explain whether each of the following numbers is prime or composite.

(a) 387 **(b)** 997

(a) Since 387 is divisible by 3, then 387 is a composite number.

(b) $\sqrt{997} = 31.6$ (to 1 d.p.), so the largest prime number less than or equal to $\sqrt{997}$ is 31.

Since 997 is not divisible by any of the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 and 31, then 997 is a prime number.

Problem-solving Tip

(a) A number is composite if it is divisible by [illegible] prime factor. (The Divisibility Test for 3 can be very helpful to check whether a big number is divisible by 3: A number is divisible by 3 if and only if the sum of the digits of the number is divisible by 3.)

(b) A number is prime if it is not divisible by *all* the prime numbers less than or equal to the square root of the number.

Are 534 and 1607 prime or composite?

6 dots can be arranged in a rectangular array in two different ways (see Fig. 1.2)

1-by-6 2-by-3

Fig. 1.2

1. Arrange each of the following numbers of dots in a rectangular array in as many different ways as possible. How many different ways are there for each number of dots?

(i) 4 dots **(ii)** 8 dots **(iii)** 12 dots **(iv)** 5 dots

(v) 7 dots **(vi)** 1 dot

2. Other than by guess and check, is there a faster method to do Question 1?

3. What do you notice about the numbers of dots that have only one arrangement? Why is this so?

4. What do you notice about the numbers of dots that have more than one arrangement? Why is this so?

Example 2

Problem involving prime number

If p and q are whole numbers such that $p \times q = 13$, find the value of $p + q$. Explain your answer.

Solution

Since 13 is a prime number, then 1 and 13 are its only two factors.

$\therefore p + q = 1 + 13 = 14$

Problem-solving Tip

It does not matter whether p (or q) is 1 or 13 because we only want to find the value of $p + q$.

Exercise 1A Questions

1. If p and q are whole numbers such that $p \times q = 31$, find the value of $p + q$. Explain your answer.
2. If n is a whole number such that $n \times (n + 28)$ is a prime number, find the prime number. Explain your answer.

Just For Fun

13 and 31 are prime numbers with reversed digits. Name another pair of prime numbers with reversed digits.

E. Index notation

We have learnt that cm² is the unit for area and cm³ is the unit for volume.

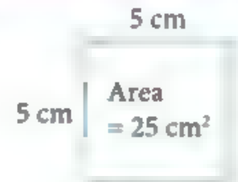


Fig. 1.3

The area of a square with length 5 cm, as shown in Fig. 1.3, is 5 cm × 5 cm = 25 cm².
5×5 can also be written as 5^2, which is read as '5 squared'.
The volume of a cube with length 5 cm, as shown in Fig. 1.4, is 5 cm × 5 cm × 5 cm = 125 cm³.
$5 \times 5 \times 5 = 5^3$, which is read as '5 cubed'.

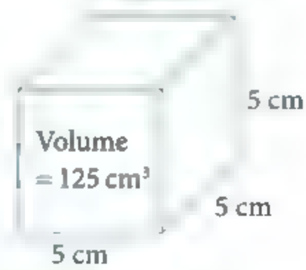


Fig. 1.4

What about $5 \times 5 \times 5 \times 5$?
We can write $5 \times 5 \times 5 \times 5$ as 5^4, which is read as '5 to the **power** of 4', where 5 is called the **base** and 4 is called the **index** (plural: **indices**).

5^4 is called the **index notation** of $5 \times 5 \times 5 \times 5$.
Write $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ in index notation:

How is the index notation useful?

For example, instead of writing that the speed of light is about 300 000 000 m/s, we can write this more concisely as 3×10^8 m/s.

The index notation is a mathematical notation used to represent the operation of 'multiplying by itself' in a *concise manner*. This ensures that we do not have to write a whole string of numbers and multiplication signs, or so many zeros.

F. Prime factorisation

Consider a composite number, e.g. 18.

It can be expressed as a product of **prime factors** as shown:

$$18 = 2 \times 3 \times 3$$
$$= 2 \times 3^2$$

The process of expressing 18 as a product of its prime factors is called the **prime factorisation** of 18.

Do not confuse the prime factorisation of 18 with finding the factors of 18:

$$18 = 1 \times 18 = 2 \times 9 = 3 \times 6.$$

Notice that the factors of 18 are 1, 2, 3, 6, 9 and 18, which are not necessarily prime factors.

Information

The Fundamental Theorem of Arithmetic states that 'every whole number greater than 1 is either a prime number or it can be expressed as a unique product of its prime factors', where there is only one product (the order of the prime factors does not matter).
In other words, prime numbers are the building blocks of whole numbers greater than 1.

Worked Example 3

Finding prime factorisation of number

Find the prime factorisation of 60, leaving your answer in index notation.

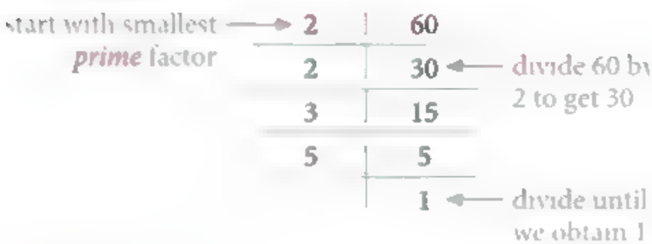
Method 1:

$\therefore 60 = 2 \times 2 \times 3 \times 5$

$= 2^2 \times 3 \times 5$

Method 2:

Divide 60 by the *smallest prime factor* and continue the process until we obtain 1.



$\therefore 60 = 2 \times 2 \times 3 \times 5$

$= 2^2 \times 3 \times 5$

Problem-solving Tip

- In practice, use a calculator to divide 60 by 2 to obtain 30, and write:
  $60 \quad 2 \times$
- Then, divide 30 by 2 to obtain 15, and write:
  $60 - 2 \times 2 \times$
- Then, divide 15 by 3 to obtain 5, and write:
  $60 = 2 \times 2 \times 3 \times 5$
- Finally, check that $2 \times 2 \times 3 \times 5$ is equal to 60.

Practice Now 3

Exercise 1A

1. Find the prime factorisation of 126, leaving your answer in index notation.
2. Express 792 as a product of its prime factors.
3. **(i)** Express 2021 as a product of its prime factors.

   (ii) Given that a and b are whole numbers such that a is less than b and $a \times b = 2021$, write down all the possible pairs (a, b).

Worked Example 4

Problem involving prime factorisation

Vasi uses 231 one-centimetre cubes to make a cuboid. Each side of the cuboid is longer than 1 cm. Find the dimensions of the cuboid.

Solution

$231 = 3 \times 7 \times 11$, where 3, 7 and 11 are prime numbers.
Since each side of the cuboid is longer than 1 cm,
then the dimensions of the cuboid are 3 cm by 7 cm by 11 cm.

Problem-solving Tip

If the length of any side of the cuboid can be 1 cm, then there is more than one possibility:

$231 \quad 3 \times 7 \times 11$
$= 1 \times 3 \times 77$
$= 1 \times 7 \times 33$
$1 \times 11 \times 21$
$= 1 \times 1 \times 231$

∴ there will be more than one answer.

Exercise 1A

1. Imran uses 195 one-centimetre cubes to make a cuboid.
   Each side of the cuboid is longer than 1 cm.
   Find the dimensions of the cuboid.
2. Joyce uses 324 one-centimetre cubes to make a cuboid.
   The perimeter of the top of the cuboid is 18 cm.
   Each side of the cuboid is longer than 2 cm. Find the height of the cuboid.

1. How are a number's prime factors different from its factors?
2. What are some methods to find the prime factorisation of a number?
3. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

Exercise 1A

1. Determine whether each of the following is a prime or a composite number.
   (a) 87 (b) 67
   (c) 73 (d) 91

2. If p and q are whole numbers such that $p \times q = 37$, find the value of $p + q$. Explain your answer.

3. If n is a whole number such that $n \times (n + 42)$ is a prime number, find the prime number. Explain your answer.

4. Find the prime factorisation of each of the following numbers, leaving your answer in index notation.
   (a) 72 (b) 756
   (c) 187 (d) 630

5. (i) Express 2026 as a product of its prime factors.
   (ii) Given that a and b are whole numbers such that a is less than b and $a \times b = 2026$, write down all the possible pairs (a, b).

6. Cheryl uses 273 one-centimetre cubes to make a cuboid. Each side of the cuboid is longer than 1 cm. Find the dimensions of the cuboid.

7. Find all the prime numbers in each of the following decades.
   (a) 2011-2020 (b) 2021-2030

8. If a and b are whole numbers such that $a \times b = 2027$, find the value of $a + b$. Explain your answer.

9. Find the prime factorisation of each of the following numbers, leaving your answer in index notation.
   (a) 8624 (b) 6804
   (c) 26 163 (d) 196 000

10. Given that x and y are whole numbers such that x is less than y and $x \times y = 2022$, write down all the possible pairs (x, y).

11. OPEN Ali wants to use 210 one-centimetre cubes to make a cuboid such that each of its sides is longer than 1 cm. There are 6 possible cuboids that he can make. Find the dimensions of any 3 of them.

12. OPEN David uses 504 one-centimetre cubes to make a cuboid. The perimeter of the top of the cuboid is 20 cm. Each side of the cuboid is longer than 2 cm. Find a possible height of the cuboid.

1.2 Square roots and cube roots

A. Squares and square roots

In Section 1.1E, we learnt that $5^2 = 5 \times 5 = 25$; we say that the *square* of 5, or 5 *squared*, is 25. The reverse is $\sqrt{25} = 5$; we say that the *square root* of 25 is 5.

Copy and complete the following:

- Since $0^2 = 0 \times 0 = 0$, then $\sqrt{0} = \sqrt{0 \times 0} = 0$.
- Since $1^2 = 1 \times 1 = \square$, then $\sqrt{1} = \sqrt{1 \times 1} = \square$.
- Since $2^2 = \square \times \square = \square$, then $\sqrt{4} = \sqrt{\square \times \square} = \square$.
- Since $3^2 = \square \times \square = \square$, then $\sqrt{\square} = \sqrt{3 \times 3} = \square$.

0, 1, 4 and 9 are squares of whole numbers, and they are called **perfect squares** (or **square numbers**). What are the next three consecutive perfect squares?

All perfect squares can be written as n^2, where the square root n is a whole number.

Worked Example 5 shows how we can find the square root of a perfect square using prime factorisation.

The superscript ^{2} represents the square of a number while the symbol $\sqrt{}$ represents the square root of a number.

Attention

The diagram below represents the inverse relationship between 'square' and 'square root':

$5^2 = 5 \times 5 = 25$

squared

5 25

square root

$\sqrt{25} = \sqrt{5 \times 5} = 5$

This diagram is useful for illustrating an inverse relationship.

Worked Example 5

Find $\sqrt{324}$ using prime factorisation.

Solution

2	324
2	162
3	81
3	27
3	9
3	3
	1

Method 1:

$$\begin{aligned} 324 &= 2 \times 2 \times 3 \times 3 \times 3 \times 3 \\ &= (2 \times 3 \times 3) \times (2 \times 3 \times 3) \\ &= (2 \times 3 \times 3)^2 \\ \therefore \sqrt{324} &= 2 \times 3 \times 3 \\ &= 18 \end{aligned}$$

Method 2:

$$\begin{aligned} 324 &= 2 \times 2 \times 3 \times 3 \times 3 \times 3 \\ &= 2^2 \times 3^4 \\ \therefore \sqrt{324} &= \sqrt{2^2 \times 3^4} \\ &= 2 \times 3^2 \\ &= 18 \end{aligned}$$

Attention

For a number to be a perfect square, the index of each prime factor must be *even*. Why?

1. Find $\sqrt{784}$ using prime factorisation.
2. Given that the prime factorisation of 7056 is $2^4 \times 3^2 \times 7^2$, find $\sqrt{7056}$ without using a calculator.
3. Use prime factors to explain why 6×24 is a perfect square.

Just For Fun

The square of the sum of the digits of a two-digit number, x, is equal to the number obtained when its digits are reversed. Find x.

B. Cubes and cube roots

$5^3 = 5 \times 5 \times 5 = 125$; we say that the **cube** of 5, or 5 **cubed**, is 125.

If we do the reverse, we get $\sqrt[3]{125} = 5$; we say that the **cube root** of 125 is 5.

What is the value of $\sqrt[3]{27}$ and of $\sqrt[3]{64}$?

Copy and complete the following:

- Since $0^3 = 0 \times 0 \times 0 = 0$, then $\sqrt[3]{0} = \sqrt[3]{0 \times 0 \times 0} = 0$.
- Since $1^3 = 1 \times 1 \times 1 = \square$, then $\sqrt[3]{1} = \sqrt[3]{1 \times 1 \times 1} = \square$.
- Since $2^3 = \square \times \square \times \square = \square$, then $\sqrt[3]{8} = \sqrt[3]{\square \times \square \times \square} = \square$.
- Since $3^3 = \square \times \square \times \square = \square$, then $\sqrt[3]{\square} = \sqrt[3]{3 \times 3 \times 3} = \square$.

0, 1, 8 and 27 are cubes of whole numbers, and they are called **perfect cubes** (or **cube numbers**). What are the next three consecutive perfect cubes?

All perfect cubes can be written as n^3, where the cube root n is a whole number.

Worked Example 6 shows how we can find the cube root of a perfect cube using prime factorisation.

The superscript ^{3} represents the cube of a number and the symbol $\sqrt[3]{}$ represents the cube root of a number.

Attention

The diagram below represents the inverse relationship between 'cube' and 'cube root'.

$5^3 = 5 \times 5 \times 5 = 125$

cubed

5 125

cube root

$\sqrt[3]{125} = \sqrt[3]{5 \times 5 \times 5} = 5$

This diagram is useful for illustrating an inverse relationship.

Worked Example 6

Finding cube root using prime factorisation

Find $\sqrt[3]{216}$ using prime factorisation.

Solution

2	216
2	108
2	54
3	27
3	9
3	3
	1

Method 1:

$$\begin{aligned} 216 &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \\ &= (2 \times 3) \times (2 \times 3) \times (2 \times 3) \\ &= (2 \times 3)^3 \end{aligned}$$

$$\begin{aligned} \therefore \sqrt[3]{216} &= 2 \times 3 \\ &= 6 \end{aligned}$$

Method 2:

$$\begin{aligned} 216 &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \\ &= 2^3 \times 3^3 \end{aligned}$$

$$\begin{aligned} \therefore \sqrt[3]{216} &= \sqrt[3]{2^3 \times 3^3} \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

Attention

For a number to be a perfect cube, the index of each prime factor must be a multiple of 3. Why?

Similar and Further

Exercise 1B

1. Find $\sqrt[3]{2744}$ using prime factorisation.
2. Given that the prime factorisation of 9261 is $3^3 \times 7^3$, find $\sqrt[3]{9261}$ without using a calculator.
3. **(i)** k is a non-zero whole number. Given that $15 \times 135 \times k$ is a perfect cube, write down the smallest value of k.

   (ii) p and q are both prime numbers. Find the values of p and q so that $15\times135\times\frac{p}{q}$ is a perfect cube.

Just For Fun

The sum of the cubes of the digits of a two-digit number, y, is equal to three times of itself. Find y.

C. Mental estimation of square roots and cube roots

What are the values of $\sqrt{50}$ and $\sqrt[3]{63}$?

Since $50 = 2 \times 5^2$ is not a perfect square (why?), $\sqrt{50}$ is not a whole number.

Similarly, since $63 = 3^2 \times 7$ is not a perfect cube (why?), $\sqrt[3]{63}$ is not a whole number.

So we cannot use the prime factorisation method to find $\sqrt{50}$ and $\sqrt[3]{63}$.

In Worked Example 7, we will learn how to **estimate** the values of numbers such as $\sqrt{50}$ and $\sqrt[3]{63}$ mentally.

7

Estimating square root and cube root

Without using a calculator, estimate the value of

(a) $\sqrt{50}$, **(b)** $\sqrt[3]{63}$.

Solution

(a) $\sqrt{50} \approx \sqrt{49} = 7$ **(b)** $\sqrt[3]{63} \approx \sqrt[3]{64} = 4$

Problem-solving Tip

(a) Find the perfect square closest to 50, which is 49. Notice that we write $\sqrt{49} = 7$, but because $\sqrt{50} \approx \sqrt{49}$, so $\sqrt{50} \approx 7$.

(b) Find the perfect cube closest to 63, which is 64.

Exercise 1B

Without using a calculator, estimate the value of

(a) $\sqrt{123}$, **(b)** $\sqrt[3]{123}$.

D. Using calculator to evaluate squares, square roots, cubes and cube roots

The following function keys on a calculator are used to find the square, square root, cube and cube root of a number.

square key   square root key

cube key

Some calculators also have the cube root key: .

For others, the cube root function can be found by pressing: .

When the evaluation involves a fraction, we can use the fraction key: .

Worked Example 8

Using calculator to evaluate square, square root, cube and cube root

Use a calculator to evaluate $\frac{8^2+\sqrt{50}}{7^3-\sqrt[3]{63}}$, leaving your answer correct to 4 decimal places.

*Solution

Method 1:

Sequence of calculator keys:

(8 x^2 + $\sqrt{\square}$ 5 0 ▶) ÷ (7 x^3 − SHIFT $\sqrt[3]{\square}$ 6 3 ▶) =

$$\frac{8^2+\sqrt{50}}{7^3-\sqrt[3]{63}} = 0.2096 \text{ (to 4 d.p.)}$$

Method 2:

Sequence of calculator keys:

$\frac{\square}{\square}$ 8 x^2 + $\sqrt{\square}$ 5 0 ▶ ▼ 7 x^3 − SHIFT $\sqrt[3]{\square}$ 6 3 ▶ =

$$\frac{8^2+\sqrt{50}}{7^3-\sqrt[3]{63}} = 0.2096 \text{ (to 4 d.p.)}$$

> **Attention**
>
> For , if you do not want to key in the first pair of brackets, you must press = before pressing − . Why?

Exercise 1B

1. Use a calculator to evaluate each of the following, leaving your answer correct to 4 decimal places where necessary.

   (a) $23^2+\sqrt{2025}-7^3$ (b) $\frac{3^2\times\sqrt{20}}{5^3-\sqrt[3]{2013}}$

2. The area of a square poster is 987 cm^2. Find the perimeter of the poster, leaving your answer correct to 1 decimal place.

3. Nadia has 2020 one-centimetre cubes. She makes the largest cube possible using some of the 2020 cubes. How many cubes does she have left over?

1. In Worked Example 5, what is something new that I have learnt about perfect squares?
2. In Worked Example 6, what is something new that I have learnt about perfect cubes?
3. What have I learnt in this section that I am still unclear of?

Exercise

1. Find each of the following using prime factorisation.
   (a) $\sqrt{1764}$ (b) $\sqrt{576}$
   (c) $\sqrt{2916}$ (d) $\sqrt{3136}$

2. Given the prime factorisation of each of the following numbers, find its square root without using a calculator.
   (a) $9801 = 3^4 \times 11^2$
   (b) $35\,721 = 3^6 \times 7^2$
   (c) $24\,336 = 2^4 \times 3^2 \times 13^2$
   (d) $518\,400 = 2^8 \times 3^4 \times 5^2$

3. Find each of the following using prime factorisation.
   (a) $\sqrt[3]{3375}$ (b) $\sqrt[3]{1728}$
   (c) $\sqrt[3]{5832}$ (d) $\sqrt[3]{8000}$

4. Given the prime factorisation of each of the following numbers, find its cube root without using a calculator.
   (a) $21\,952 = 2^6 \times 7^3$
   (b) $46\,656 = 2^6 \times 3^6$
   (c) $287\,496 = 2^3 \times 3^3 \times 11^3$
   (d) $1\,728\,000 = 2^9 \times 3^3 \times 5^3$

5. Without using a calculator, estimate the value of each of the following.
   (a) $\sqrt{66}$ (b) $\sqrt{80}$
   (c) $\sqrt[3]{218}$ (d) $\sqrt[3]{730}$

6. Use a calculator to evaluate each of the following, leaving your answer correct to 4 decimal places where necessary.
   (a) $7^2 - \sqrt{361} + 21^3$ (b) $\dfrac{\sqrt{555} + 5^2}{2^3 \times \sqrt[3]{222}}$
   (c) $\sqrt{4^3 + \sqrt[3]{4913}}$

7. Find the smallest non-zero whole number which can be multiplied by 112 to give a square number.

8. Find the smallest non-zero whole number which can be multiplied by 162 to give a cube number.

9. A textbook is opened at random. Without using a calculator, find the pages the textbook is opened to, given that the product of the facing numbers is 420.
   Hint: 400 is a perfect square.

10. The area of a square photo frame is 250 cm². Find the perimeter of the photo frame, leaving your answer correct to 1 decimal place.

11. The volume of a box in the shape of a cube is 2197 cm³. Find the area of one side of the box.

12. Raju has 2020 one-centimetre square tiles. He makes the largest square possible using some of the 2020 square tiles. How many square tiles does he have left over?

13. Use prime factors to explain why 6×54 is a perfect square.

14. (i) k is a non-zero whole number. Given that $6 \times 54 \times k$ is a perfect cube, write down the smallest value of k.
    (ii) p and q are both prime numbers. Find the values of p and q so that $6 \times 54 \times \dfrac{p}{q}$ is a perfect cube.

1.3 Highest common factor and lowest common multiple

A. Highest common factor (HCF)

In Section 1.1A, we learnt how to find the factors of a number. What are the factors of 18 and 30?

$$\begin{aligned} 18 &= 1 \times 18 \\ &= 2 \times 9 \\ &= 3 \times 6, \end{aligned} \qquad \begin{aligned} 30 &= 1 \times 30 \\ &= 2 \times 15 \\ &= 3 \times 10 \\ &= 5 \times 6. \end{aligned}$$

Factors of 18:	1,	2,	3,		6,	9,			18	
Factors of 30:	1,	2,	3,	5,	6,		10,	15,		30

From the list above, we see that 1, 2, 3 and 6 are the **common factors** of 18 and 30.

Of all the common factors of 18 and 30, the highest is 6.

We say that the **highest common factor (HCF)** of 18 and 30 is 6.

This method of finding the HCF of two or more non-zero whole numbers is called the **listing method**.

> **Attention**
> The lowest common factor of 18 and 30 is 1. In fact, the lowest common factor of any two or more non-zero whole numbers is always 1.

What is the HCF of 504 and 588?

The listing method to find the HCF of numbers like 504 and 588 is tedious because it involves many factors and common factors.

We will now learn more efficient methods to find the HCF of two numbers.

Worked Example 9

Finding HCF of two numbers

Find the highest common factor of 18 and 30.

Solution

Method 1 Prime factorisation

common prime factors

$$\begin{aligned} 18 &= \boxed{2} \times \boxed{3} \times 3 \\ 30 &= \boxed{2} \times \boxed{3} \times 5 \end{aligned}$$

HCF of 18 and 30 $= 2 \times 3$
$= 6$

or

common prime factor

$$\begin{aligned} 18 &= \boxed{2} \times \boxed{3^2} \\ 30 &= \boxed{2} \times \boxed{3} \times 5 \end{aligned}$$

2×3 ← common factor is 3, i.e. choose 3 with the *smaller index*

Method 2 Ladder method

common prime factors

$\boxed{2}$	18, 30	← divide 18 and 30 by 2 to get 9 and 15
$\boxed{3}$	9, 15	← divide 9 and 15 by 3 to get 3 and 5
	3, 5	← stop dividing when there are no common prime factors

HCF of 18 and 30 $= 2 \times 3$
$= 6$

1. Find the highest common factor of 56 and 84 using both methods.
2. Using the prime factorisation method, find the largest whole number that is a factor of both 112 and 140.
3. The numbers 504 and 588, written as the products of their prime factors, are $504 = 2^3 \times 3^2 \times 7$ and $588 = 2^2 \times 3 \times 7^2$. Hence, explain why 84 is the greatest whole number that will divide both 504 and 588 exactly.

Problem-solving Tip

2. The largest whole number that is a factor of both 112 and 140 is the HCF of 112 and 140.

B. Lowest common multiple (LCM)

In primary school, we have learnt how to find multiples, e.g.

Multiples of 4:	4,		8,	12,	16,		20,	24,	28,		32,	36,	40,	...
Multiples of 6:		6,		12,		18,		24,		30,		36,		42, ...

From the list, we see that 12, 24, 36, ... are the **common multiples** of 4 and 6.

How many common multiples of 4 and 6 are there?

What is the highest common multiple of 4 and 6?

Of all the common multiples of 4 and 6, the lowest is 12, i.e. the **lowest common multiple (LCM)** of 4 and 6 is 12.

This method of finding the LCM of two non-zero whole numbers is called the **listing method**.

Attention

The highest common multiple of any two or more non-zero whole numbers is always undefined.

What is the LCM of 504 and 540?

It can be very tedious to list the multiples of numbers like 504 and 540 to find the LCM.

We will now learn more efficient methods to find the LCM of two numbers.

Finding LCM

The product of 4 and 6, i.e. 24, is a common multiple of 4 and 6. However, using the listing method, we found that the LCM of 4 and 6 is 12 and not 24. Why is this so?

Since the LCM of 4 and 6 is a multiple of 4 and a multiple of 6, we get:

$$4 \times h = 6 \times k$$
$$2 \times 2 \times h = 2 \times 3 \times k$$
$$2 \times 2 \times \underset{\boxed{3}}{\underset{\downarrow}{h}} = 2 \times \underset{\boxed{2}}{\underset{\downarrow}{k}} \times 3$$

Both sides of the equation contain a prime factor 2 (highlighted in green).

We can see that for both sides of the equation to be equal, $h = 3$ (highlighted in purple) and $k = 2$ (highlighted in blue).

$\therefore$ LCM of 4 and 6 $= 2 \times 2 \times 3$.

From the above explanation, we derive the **prime factorisation** method of finding the LCM of 4 and 6 as shown below:

common prime factor: take only one

$$
\begin{aligned}
4 &= \boxed{2} \times 2 \\
6 &= \boxed{2} \times \times 3 \\
\text{LCM of 4 and 6} &= 2 \times 2 \times 3 \\
&= 12
\end{aligned}
$$

or

choose the common factor (i.e. 2) with the *higher index* (why?)

$$
\begin{aligned}
4 &= \boxed{2^2} \\
6 &= \boxed{2} \times 3 \\
& 2^2 \times 3
\end{aligned}
$$

The **ladder method** for finding the LCM of 4 and 6 is as follows:

$$
\begin{array}{r|ll}
\boxed{2} & 4, & 6 \\
\hline
 & \boxed{2,} & \boxed{3}
\end{array}
$$

common prime factor → $\boxed{2}$

4, 6 ← divide 4 and 6 by 2 to get 2 and 3

2, 3 ← stop dividing when there are no common prime factors

LCM of 4 and 6 = $2 \times \underbrace{2 \times 3}_{\text{remaining factors}}$

= 12

Example 10

Finding LCM of two numbers

Find the lowest common multiple of 30 and 36.

Solution

Method 1: Prime factorisation

common prime factors

$$
\begin{aligned}
30 &= \boxed{2} \times \boxed{3} \times 5 \\
36 &= \boxed{2} \times 2 \times \boxed{3} \times 3 \\
\text{LCM of 30 and 36} &= 2 \times 2 \times 3 \times 3 \times 5 \\
&= 180
\end{aligned}
$$

or

choose each of the common prime factors (i.e. 2 and 3) with the *higher index* and the remaining factor (i.e. 5)

$$
\begin{aligned}
30 &= \boxed{2} \times \boxed{3} \times 5 \\
36 &= \boxed{2^2} \times \boxed{3^2} \\
& 2^2 \times 3^2 \times 5
\end{aligned}
$$

Method 2: Ladder method

$$
\begin{array}{r|ll}
\boxed{2} & 30, & 36 \\
\hline
\boxed{3} & 15, & 18 \\
\hline
 & \boxed{5,} & \boxed{6}
\end{array}
$$

prime factors

15, 18 ← divide 15 and 18 by 3 to get 5 and 6

5, 6 ← stop dividing when there are no common prime factors

LCM of 30 and 36 = $2 \times 3 \times \underbrace{5 \times 6}_{\text{remaining factors}}$

= 180

Practise Now 10

Exercise 1C

1. Find the lowest common multiple of 24 and 90 using both methods.
2. The numbers 120 and 126, written as the products of their prime factors, are $120 = 2^3 \times 3 \times 5$ and $126 = 2 \times 3^2 \times 7$. Hence, find the smallest non-zero whole number that is divisible by both 120 and 126.

Worked Example 11

Finding factors of number

A number has exactly 8 factors, two of which are 6 and 27. List all the factors of the number.

Solution

$$6 = 2 \times 3$$
$$27 = 3^3$$
$$\text{LCM of 6 and 27} = 2 \times 3^3 = 54$$

$$54 = 1 \times 54 = 2 \times 27 = 3 \times 18 = 6 \times 9$$

$\therefore$ the number is 54 and its factors are 1, 2, 3, 6, 9, 18, 27 and 54.

> **Problem-solving Tip**
>
> List the factors of the LCM of 6 and 27 to see if there are 8 factors. Since there are 8 factors, then the number is the LCM of 6 and 27.
> If there are less than 8 factors, try multiplying the LCM by a prime number to see if the result has exactly 8 factors (see Practise Now 11 Question 2).
> Sometimes, it is possible to obtain more than one answer (see Exercise 1C Question 17).

Practise Now 11

1. A number has exactly 8 factors, two of which are 27 and 45. List all the factors of the number.
2. A number has exactly 8 factors, two of which are 4 and 20. List all the factors of the number.

Worked Example 12

Finding number given another number and their LCM

Find the smallest value of n such that the LCM of n and 6 is 24.

Solution

$\text{LCM} = 24 = 2 \times 2 \times 2 \times 3$ $\qquad$ $\text{LCM} = 24 = 2 \times 2 \times 2 \times 3$

$\therefore n = 2 \times 2 \times 2 = 8$

> **Problem-solving Tip**
>
> This is a reverse question, so use the prime factorisation method backwards.
> There is at most one common prime factor between n and 6.
> The first factor 2 for n will be common with the factor 2 for 6.
> We do not use factor 3 since we want n to be the smallest.
> The other two factors (2 and 2) must be the factors of n.

Practise Now 12

Similar and Further Questions
Exercise 1C
Question

Find the smallest value of n such that the LCM of n and 15 is 45.

Worked Example 13

Finding numbers given their HCF and LCM

The highest common factor of two numbers is 56.
The lowest common multiple of these two numbers is 2520.
Both numbers are greater than 56. Find the two numbers.

We will use Polya's Problem Solving Model to guide us in solving this problem.

Stage 1: Understand the problem

This is a reverse question because the two numbers are not given but their HCF and LCM are given.
Both numbers are also given to be greater than their HCF, so neither of them can be equal to the HCF (= 56).

Stage 2: Think of a plan

Since the HCF is a factor of each of the two numbers, then both numbers must contain the HCF as a factor as shown:
Let the two numbers be a and b.

$$a = \boxed{2^3} \times \boxed{7} \times ?$$
$$b = \boxed{2^3} \times \boxed{7} \times ?$$

$$\text{HCF} = 56 = 2^3 \times 7$$
$$\text{LCM} = 2520 = 2^3 \times 7 \times 3^2 \times 5$$

Since the LCM contains three more factors (i.e. 3, 3 and 5), we have to distribute these factors into a and b such that:

- the HCF is still $2^3 \times 7$ (i.e. we cannot 'give' one 3 to a and the other 3 to b, or else the HCF will become $2^3 \times 7 \times 3$);
- both a and b are greater than the HCF (i.e. we cannot 'give' all the remaining factors to only a or b; since we 'give' 3^2 to one of them, we have to give 5 to the other number).

Stage 3: Carry out the plan

HCF $= 56 = 2^3 \times 7$
LCM $= 2520 = 2^3 \times 3^2 \times 5 \times 7 = (2^3 \times 7) \times 3^2 \times 5$
Let the two numbers be a and b.

$$a = \boxed{2^3} \times \boxed{7} \times 3^2$$
$$b = \boxed{2^3} \times \boxed{7} \qquad \times 5$$

$$\text{HCF} = 56 = 2^3 \times 7$$
$$\text{LCM} = 2520 = 2^3 \times 7 \times 3^2 \times 5$$

$\therefore$ the two numbers are $2^3 \times 7 \times 3^2 = 504$ and $2^3 \times 7 \times 5 = 280$.

Problem-solving Tip

It does not matter whether we 'give' 3^2 to a or to b.

Stage 4: **Look back**

How can we check that the answer is correct?

Method 1: Find the HCF and LCM of 280 and 504.

Method 2: Use the fact that the product of the two numbers a and b, is equal to the product of their HCF and LCM, i.e. $a \times b = \text{HCF} \times \text{LCM}$ (why?).

Check: $280 \times 504 = 141\ 120$

$\text{HCF} \times \text{LCM} = 56 \times 2520 = 141\ 120$

Attention

The product of 3 numbers may not be equal to HCF × LCM

Practise Now 13

Exercise 1

1. The highest common factor of two numbers is 245.
The lowest common multiple of these two numbers is 4410.
Both numbers are greater than their highest common factor.
Find the two numbers.

2. The numbers 240 and 252, written as the products of their prime factors, are $240 = 2^4 \times 3 \times 5$ and $252 = 2^2 \times 3^2 \times 7$. Find
   (i) the smallest non-zero whole number n for which $240n$ is a multiple of 252,
   (ii) the smallest non-zero whole number m for which $\frac{240}{m}$ is a factor of 252.

C. Real-life applications of HCF and LCM

We have learnt how prime numbers can help us find the HCF and LCM of two numbers. In this section, we will solve some real-life problems involving HCF and LCM.

Worked Example 14

Real-life problem involving LCM

The lights on two lightships flash at regular intervals. The first light flashes every 18 seconds and the second every 40 seconds. The two lights flash together at 10.00 p.m. At what time will they next flash together?

*Solution

$18 = 2 \times 3^2$

$40 = 2^3 \times 5$

$\therefore$ LCM of 18 and 40 $= 2^3 \times 3^2 \times 5$

$= 360$

360 seconds = 6 minutes

$\therefore$ the two lights will next flash together at 10.06 p.m.

Problem-solving Tip

The LCM of the two timings is the interval at which the lights flash together.

Practise Now 14

Similar and Further Questions

Exercise 1C

1. Two bells toll at regular intervals of 15 minutes and 36 minutes respectively. If they toll together at 2.00 p.m., what time will they next toll together?

2. Li Ting has two pieces of rope measuring 140 cm and 168 cm. She wishes to cut the two pieces of rope equally into smaller pieces without any leftover rope.
   (i) What is the greatest possible length of each of the smaller pieces of rope?
   (ii) How many smaller pieces of rope can she cut altogether?

Introductory Problem Revisited

Now that you have learnt about HCF and LCM, solve the and discuss your solution with your classmates.

1. What do I already know about common factors and common multiples that could help me understand what HCF and LCM are?
2. When faced with a real-life problem involving HCF or LCM, how do I tell whether to use HCF or LCM to find the solution?

Basic Intermediate

Exercise

1. Find the highest common factor of each of the following sets of numbers.
   (a) 12 and 30 (b) 13 and 91
   (c) 126 and 240 (d) 180 and 450
   (e) 11 and 31 (f) 64 and 81

2. Find the largest whole number that is a factor of both 156 and 168

3. Find the lowest common multiple of each of the following sets of numbers.
   (a) 45 and 60 (b) 42 and 462
   (c) 54 and 240 (d) 11 and 19
   (e) 27 and 32 (f) 78 and 352

4. Explain why 1040 is the smallest non-zero whole number that is divisible by both 80 and 104.

5. A number has exactly 8 factors, two of which are 10 and 125. List all the factors of the number.

6. A number has exactly 12 factors, two of which are 40 and 100. List all the factors of the number.

7. Find the smallest value of k such that the LCM of k and 6 is 60.

8. The numbers 792 and 990, written as the products of their prime factors, are $792 = 2^3 \times 3^2 \times 11$ and $990 = 2 \times 3^2 \times 5 \times 11$. Hence, explain why 198 is the greatest whole number that will divide both 792 and 990 exactly.

9. The numbers 176 and 342, written as the products of their prime factors, are $176 = 2^4 \times 11$ and $342 = 2 \times 3^2 \times 19$. Hence, find the smallest non-zero whole number that is divisible by both 176 and 342.

10. A number has exactly 8 factors, two of which are 4 and 26. List all the factors of the number.

11. The highest common factor of two numbers is 175. The lowest common multiple of these two numbers is 12 600.
    Both numbers are greater than their highest common factor.
    Find the two numbers.

12. (i) Express 1050 as the product of its prime factors.
    (ii) Find two numbers, both greater than 40, that have a highest common factor of 21 and a lowest common multiple of 1050.

Intermediate Basic

Exercise 1C

13. Shaha needs to pack 171 pens and 63 pencils equally into identical gift bags. Find
(i) the largest number of gift bags that can be packed,
(ii) the number of each item in a gift bag.

14. Sweets are sold in packs of 120 while mini chocolate bars are sold in packs of 18.
Sara bought the same number of sweets as mini chocolate bars.
Find the least number of packs of sweets that she could have bought.

15. Two race cars, Car X and Car Y, are at the starting point of a 2-km track at the same time Car X and Car Y make one lap every 60 s and every 80 s respectively.
(i) How long, in seconds, will it take for both cars to be back at the starting point at the same time?
(ii) How long, in minutes, will it take for the faster car to be 5 laps ahead of the slower car?

16. Ken wishes to cut the biggest possible squares from a rectangular sheet of paper without any leftover paper. The sheet of paper has a length of 65 cm and a breadth of 50 cm.
(i) What is the length of each square?
(ii) How many squares can he cut altogether?

17. OPEN Find two numbers that each have exactly 16 factors, two of which are 8 and 12.

18. The numbers 528 and 540, written as the products of their prime factors, are $528 = 2^4 \times 3 \times 11$ and $540 = 2^2 \times 3^3 \times 5$. Hence, find
(i) the smallest non-zero whole number h for which $528h$ is a multiple of 540,
(ii) the smallest whole number k for which $\frac{528}{k}$ is a factor of 540.

19. The numbers 630 and 1248, written as the products of their prime factors, are $630 = 2 \times 3^2 \times 5 \times 7$ and $1248 = 2^5 \times 3 \times 13$. Find
(i) the smallest non-zero whole number n for which $630n$ is a multiple of 1248,
(ii) the smallest whole number m for which $\frac{1248}{m}$ is a factor of 630.

20. A class has between 30 to 40 students. Each boy in the class brings 15 chocolate bars for a class party The chocolate bars are shared equally among the 20 girls of the class and their form teacher. There are no leftovers.
(i) How many students are there in the class?
(ii) How many chocolate bars does their form teacher receive?

In this chapter, we learnt about prime numbers, the building blocks of whole numbers greater than 1. We also learnt that whole numbers greater than 1 can be classified into two categories: prime numbers and composite numbers.

Here are a few important lessons you must remember as we move forward:

1. Definitions in mathematics are important because they allow us to specify the properties of mathematical objects, such as prime numbers We must learn to understand and apply them as we examine new mathematical objects.
2. Index notations help us write the prime factorisation of a number more concisely. Notations are important because they help us represent mathematical objects and their operations or relationships in a more concise and precise manner.

3. We can use the prime factorisation of two numbers to find their highest common factor (HCF) and lowest common multiple (LCM). There are many useful applications of primes, HCF and LCM, both in mathematics and in the real world.

1. A **prime number** is a whole number that has *exactly 2 different factors*, 1 and itself.
   Examples of prime numbers are 2 and 7.
   - Give two other examples.
2. (a) A **composite number** is a whole number that has *more than 2 different factors*.
   Examples of composite numbers are 6 and 15.
   - Give two other examples.

   (b) The process of expressing a composite number as a product of its prime factors is known as **prime factorisation**, e.g. $18 = 2 \times 3 \times 3$ (or 2×3^2).
   - Give another example of prime factorisation.
3. Whole numbers can be classified into three groups as shown below.
   Fill in the blanks.

Whole numbers (0, 1, 2, 3, 4, ...)

Neither prime nor composite	**Prime numbers**	**Composite numbers**
	has exactly 2 different factors	has more than 2 different factors
(and)	(2, 3, 5, 7, 11, ...).	(4, 6, 8, 9, 10, ...).

4. A **perfect square** (or square number) is a whole number whose **square root** is also a whole number.
   An example of a perfect square is 25 because $25 = 5^2$, or $\sqrt{25} = 5$.
   - Give two other examples of perfect squares and find their square roots.
5. A **perfect cube** (or cube number) is a whole number whose **cube root** is also a whole number.
   An example of a perfect cube is 125 because $125 = 5^3$, or $\sqrt[3]{125} = 5$.
   - Give two other examples of perfect cubes and find their cube roots.
6. The **highest common factor (HCF)** of two numbers is the largest factor that is common to all the numbers, e.g. the HCF of 18 and 30 is 6.
   - Give another example.
7. The **lowest common multiple (LCM)** of two numbers is the smallest multiple that is common to all the numbers, e.g. the LCM of 18 and 56 is 504.
   - Give another example.

   Think of a real-life problem that you can use the HCF or the LCM to help you solve.

CHAPTER 2

Fractions

Ancient Egyptians had a developed fraction system dated 4000 years ago. Unlike the common fractions that we use today, the ancient Egyptians used symbols to write unit fractions, where the numerator is 1. Other fractions (except $\frac{2}{3}$ and $\frac{3}{4}$) were written as a sum of unit fractions, with each unit fraction appearing only once in the sequence. For example, $\frac{3}{5}$ can be expressed as $\frac{1}{2}+\frac{1}{10}$. It can also be expressed as $\frac{1}{2}+\frac{1}{20}+\frac{1}{30}+\frac{1}{60}$.

Are you able to verify that $\frac{1}{2}+\frac{1}{10}$ and $\frac{1}{2}+\frac{1}{20}+\frac{1}{30}+\frac{1}{60}$ are both equal to $\frac{3}{5}$?

Besides addition, we will learn how to perform other basic operations on proper fractions, improper fractions and mixed numbers in this chapter.

Learning Outcomes

What will we learn in this chapter?

- What equivalent fractions are and how to compare them
- What mixed numbers and improper fractions are
- How to perform basic operations involving fractions and mixed numbers

Introductory Problem

A baker wishes to make chocolate chip cookies based on the recipe in Fig. 2.1.

If he had 2 cups of white sugar left in his kitchen, determine

(a) the number of cookies he can make, and

(b) the amount of each of the other ingredients he needs.

Recipe for chocolate chip cookies
(for 12 cookies)

- $\frac{1}{4}$ cup butter
- $\frac{1}{4}$ cup white sugar
- $\frac{1}{4}$ cup brown sugar
- $\frac{1}{2}$ egg
- $\frac{1}{2}$ teaspoon vanilla extract
- $\frac{1}{4}$ teaspoon baking soda
- $\frac{3}{4}$ cups flour
- 55 g chocolate chips

Fig. 2.1

Discuss with your classmates your method for part **(a)**. What are some ways of solving the same question?

In this chapter, we will learn how to solve such problems involving operations on fractions and mixed numbers.

2.1 Fractions, improper fractions and mixed numbers

A. Equivalent fractions

In primary school, we have learnt **proper fractions**, which are less than 1, such as

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6} \text{ and } \frac{9}{12}$$

Fig. 2.2

Which two fractions have the same value? ☐ and ☐

These two fractions are **equivalent**.

Equivalent fractions are useful when we want to express a fraction in its simplest form or to compare fractions.

Equivalent fractions have the same value, e.g. $\frac{1}{5}$ and $\frac{2}{10}$.
To perform some operations, we have to convert a fraction from one form to another equivalent form, e.g.
$\frac{1}{5} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$

Worked Example 1

Expressing a fraction in its simplest form

Express $\frac{9}{12}$ in its simplest form.

Solution

$\div 3$

$$\frac{9}{12}=\frac{3}{4}$$

Problem-solving Tip

To simplify a fraction, we divide the numerator and denominator by their [illegible].

Practice Now 1

Identify the fractions which are in the simplest form and simplify those that are not.

$\frac{1}{2}$ $\frac{3}{4}$ $\frac{4}{6}$ $\frac{5}{7}$ $\frac{6}{9}$ $\frac{8}{10}$ $\frac{13}{23}$ $\frac{20}{200}$

Worked Example 2

Comparing fractions with different denominators

Compare the following pairs of fractions by filling the blanks with '>', '<' or '='.

(a) $\frac{5}{6}$ ☐ $\frac{9}{12}$ (b) $\frac{5}{9}$ ☐ $\frac{7}{12}$

Notations, such as >, < and =, help to present the relationships between numbers concisely and precisely.

Solution

(a) From Fig. 2.2,

$\frac{5}{6}$

$\frac{9}{12}$

From the figure above,

$$\frac{5}{6}=\frac{10}{12}$$ LCM of the denominators of $\frac{5}{6}$ and $\frac{9}{12}$ is 12

$\times 2$

$$\frac{10}{12}>\frac{9}{12} \quad \therefore \frac{5}{6}>\frac{9}{12}$$

Problem-solving Tip

Since both $\frac{10}{12}$ and $\frac{9}{12}$ have the same denominator, we compare their numerators.

$\times 4$ $\times 3$

(b) $\frac{5}{9}=\frac{20}{36}$ and $\frac{7}{12}=\frac{21}{36}$

$\times 4$ $\times 3$

$$\frac{20}{36}<\frac{21}{36} \quad \therefore \frac{5}{9}<\frac{7}{12}$$

Exercise 2A

1. Compare the following pairs of fractions by filling the blanks with '>', '<' or '='.

(a) $\frac{2}{3}$ ____ $\frac{7}{9}$ (b) $\frac{3}{4}$ ____ $\frac{7}{10}$

(c) $\frac{9}{21}$ ____ $\frac{27}{63}$ (d) $\frac{2}{5}$ ____ $\frac{3}{8}$

2. Convert the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{9}$ and $\frac{7}{12}$ to their equivalent forms that have the same denominator. Hence arrange the fractions in order, starting with the smallest.

B. Mixed numbers and improper fractions

A **mixed number** consists of a whole number and a proper fraction.
For example,

1 $\frac{1}{6}$

Attention

In some countries, the term '*mixed fractions*' is used to describe 'mixed numbers'.

may be represented as a *mixed number* $1\frac{1}{6}$.

It can also be expressed as an **improper fraction**, in which the numerator is *greater than* the denominator.

$\frac{6}{6}$ $\frac{1}{6}$

$$1\frac{1}{6} = \frac{6}{6} + \frac{1}{6}$$
$$= \frac{7}{6}$$

Mixed numbers and improper fractions can also be simplified using what we have learnt in Section 2.1A.

Worked Example 3

Expressing mixed number as improper fraction

Express $2\frac{4}{12}$ as an improper fraction in its simplest form.

Solution

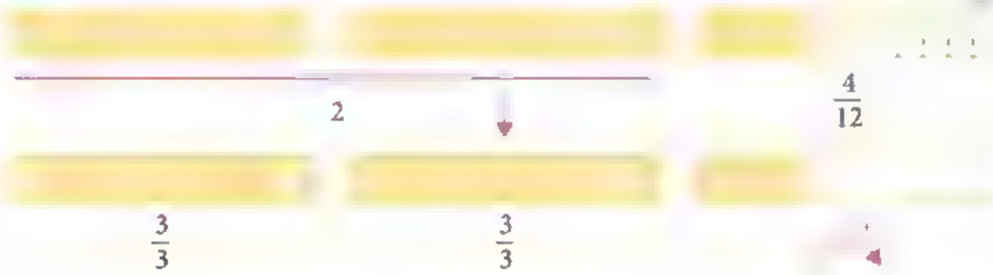
Method 1:

$$2\frac{4}{12} = \frac{12}{12} + \frac{12}{12} + \frac{4}{12}$$
$$= \frac{28}{12}$$
$$= \frac{7}{3} \quad \text{reduce to the simplest form}$$

Attention

Always express your answer in the *simplest form* (or lowest terms), e.g. $\frac{7}{3}$ and not $\frac{28}{12}$.

Method 2



$$\frac{4}{12} = \frac{1}{3}$$

$$2\frac{4}{12} = 2\frac{1}{3}$$

$$= \frac{3}{3} + \frac{3}{3} + \frac{1}{3}$$

$$= \frac{7}{3}$$

Reflection

How does Method 2 differ from Method 1? Which method do you prefer? Why?

Practise Now 3

Exercise 2A

Express each of the following as an improper fraction in its simplest form

(a) $2\frac{2}{3}$ **(b)** $5\frac{8}{10}$

Worked Example 4

Expressing improper fraction as mixed number

Express $\frac{15}{6}$ as a mixed number in its simplest form.

Solution

$$\frac{15}{6} = \frac{5}{2}$$ simplify the improper fraction

$$= \frac{4}{2} + \frac{1}{2}$$

$$= 2\frac{1}{2}$$

Practise Now 4

Exercise 2A

Express each of the following as a mixed number in its simplest form.

(a) $\frac{27}{4}$ **(b)** $\frac{30}{9}$

Comparing mixed number and improper fraction

Compare the following pairs of numbers using '>', '<' or '='.

(a) $1\frac{1}{2}$ ☐ $1\frac{5}{12}$ (b) $\frac{8}{3}$ ☐ $\frac{16}{5}$ (c) $\frac{13}{5}$ ☐ $2\frac{7}{10}$

Solution

(a) Since the whole numbers in $1\frac{1}{2}$ and $1\frac{5}{12}$ are the same, we compare the proper fractions

$\times 6$

$\frac{1}{2} = \frac{6}{12}$ and $\frac{5}{12}$.

$\times 6$

Since $\frac{6}{12} > \frac{5}{12}$, thus $\frac{1}{2} > \frac{5}{12}$.

$\therefore 1\frac{1}{2} > 1\frac{5}{12}$

(b) **Method 1:**

$\times 5$ $\times 3$

$\frac{8}{3} = \frac{40}{15}$ and $\frac{16}{5} = \frac{48}{15}$

$\times 5$ $\times 3$

Since $\frac{40}{15} < \frac{48}{15}$, thus $\frac{8}{3} < \frac{16}{5}$.

Method 2:

$\frac{8}{3} = \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$

$= 2\frac{2}{3}$

$\frac{16}{5} = \frac{5}{5} + \frac{5}{5} + \frac{5}{5} + \frac{1}{5}$

$= 3\frac{1}{5}$

Since $2 < 3$, thus $2\frac{2}{3} < 3\frac{1}{5}$.

$\therefore \frac{8}{3} < \frac{16}{5}$.

(c) **Method 1:**

$\frac{13}{5} = 2\frac{3}{5}$

$\times 2$

$\frac{3}{5} = \frac{6}{10}$ and $\frac{7}{10}$

$\times 2$

Since $\frac{6}{10} < \frac{7}{10}$, thus $\frac{3}{5} < \frac{7}{10}$.

$\therefore \frac{13}{5} < 2\frac{7}{10}$.

Method 2:

$\times 2$

$\frac{13}{5} = \frac{26}{10}$ and $2\frac{7}{10} = \frac{27}{10}$

$\times 2$

Since $\frac{26}{10} < \frac{27}{10}$, thus $\frac{13}{5} < \frac{27}{10}$.

$\therefore \frac{13}{5} < 2\frac{7}{10}$.

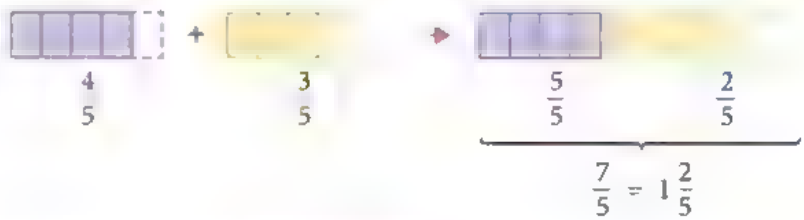
Compare the following pairs of numbers using '>', '<' or '='.

(a) $2\frac{2}{5}$ ☐ $2\frac{12}{30}$ (b) $\frac{37}{15}$ ☐ $\frac{14}{5}$ (c) $\frac{26}{7}$ ☐ $3\frac{3}{5}$

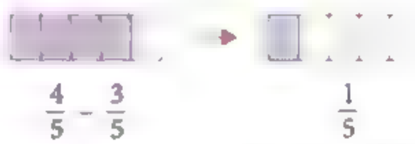
1. How did what I have learnt about lowest common multiple help me in finding equivalent fractions?
2. What have I learnt in this section that I am still unclear of?

2.2 Adding and subtracting fractions and mixed numbers

In primary school, we learnt to add and subtract fractions.



$\frac{4}{5}$ and $\frac{3}{5}$ have the same denominator. They are **like fractions**.
To add like fractions, we add the numerators.
Similarly, to subtract like fractions, we subtract the numerators.



To add or subtract **unlike fractions**, which have different denominators such as $\frac{3}{5}$ and $\frac{1}{10}$, we need to first express them in the same denominator. Worked Example 6 shows this.

Worked Example 6

Adding and subtracting fractions

Without using a calculator, evaluate

(a) $\frac{1}{2}+\frac{3}{4}$, (b) $\frac{2}{3}-\frac{1}{2}$.

Write each answer in its simplest form, where necessary.

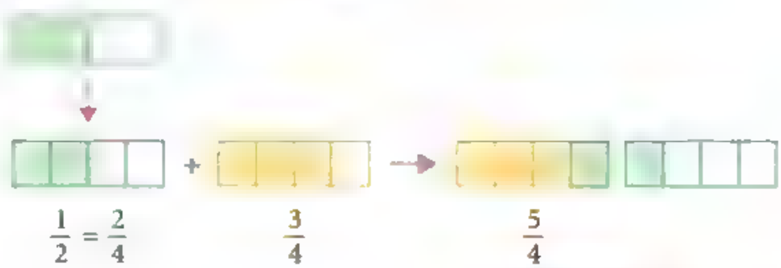
Solution

(a) $\frac{1}{2}+\frac{3}{4}=\frac{2}{4}+\frac{3}{4}$

$=\frac{2+3}{4}$

$=\frac{5}{4}$

$=1\frac{1}{4}$



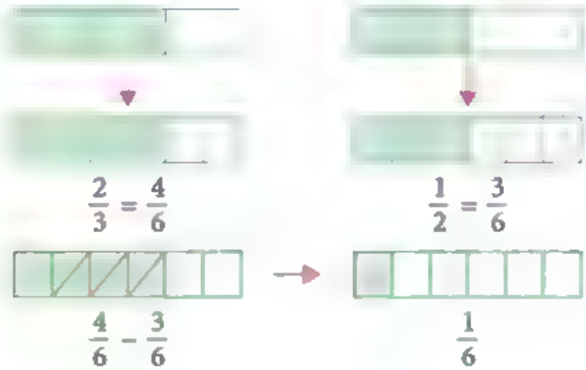
Attention

While it is not wrong to leave your numerical answers in improper fractions, it is generally neater to leave them in *mixed numbers*.

(b) $\dfrac{2}{3} - \dfrac{1}{2} = \dfrac{4}{6} - \dfrac{3}{6}$

$= \dfrac{4-3}{6}$

$= \dfrac{1}{6}$



Exercise 2A

1. Evaluate each of the following without using a calculator. Write each answer in its simplest form, where necessary.

(a) $\dfrac{2}{13} + \dfrac{6}{13}$ (b) $\dfrac{1}{3} + \dfrac{5}{9}$

(c) $\dfrac{3}{4} + \dfrac{5}{6}$ (d) $\dfrac{7}{2} + \dfrac{11}{6}$

2. Evaluate each of the following without using a calculator. Write each answer in its simplest form, where necessary.

(a) $\dfrac{11}{15} - \dfrac{7}{15}$ (b) $2 - \dfrac{3}{4}$

(c) $\dfrac{4}{5} - \dfrac{1}{4}$ (d) $\dfrac{15}{4} - \dfrac{5}{9}$

Worked Example 7

Adding and subtracting mixed numbers

Evaluate each of the following without using a calculator. Write each answer as a mixed number in its simplest form, where necessary.

(a) $1\dfrac{3}{8} + 2\dfrac{1}{8}$ (b) $2\dfrac{1}{4} + 3\dfrac{7}{8}$

(c) $2\dfrac{7}{10} - 1\dfrac{3}{10}$ (d) $2\dfrac{1}{5} - 1\dfrac{2}{3}$

Solution

(a)

$1\dfrac{3}{8}$ $2\dfrac{1}{8}$

$1\dfrac{3}{8} + 2\dfrac{1}{8} = 3\dfrac{3+1}{8}$

$= 3\dfrac{4}{8}$

$= 3\dfrac{1}{2}$ reduce to the simplest form

(b) $2\frac{1}{4} + 3\frac{7}{8} = 2\frac{2}{8} + 3\frac{7}{8}$

$= 5\frac{2+7}{8}$

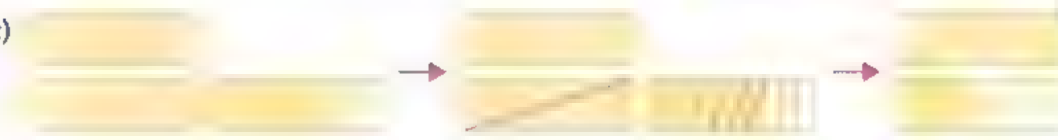
$= 5\frac{9}{8}$

$= 6\frac{1}{8}$

Problem-solving tip

When we add or subtract [illegible] in Worked Example 6, we first convert them to [illegible] Similarly, when we add mixed numbers with differing denominators, we first express them in the same denominator.

(c)



$2\frac{7}{10}$ $\qquad$ $2\frac{7}{10} - 1\frac{3}{10}$

$2\frac{7}{10} - 1\frac{3}{10} = 1\frac{7-3}{10}$

$= 1\frac{4}{10}$

$= 1\frac{2}{5}$ $\qquad$ reduce to the simplest form

(d) $2\frac{1}{5} - 1\frac{2}{3}$ $\qquad$ LCM of the denominators of $\frac{1}{5}$ and $\frac{2}{3}$ is 15

$= 2\frac{3}{15} - 1\frac{10}{15}$

$= 1\frac{18}{15} - 1\frac{10}{15}$

$= \frac{18-10}{15}$

$= \frac{8}{15}$

1. Evaluate each of the following without using a calculator. Write each answer in its simplest form, where necessary.

(a) $2\frac{2}{15} + 1\frac{8}{15}$ $\qquad$ **(b)** $5\frac{3}{11} + 2\frac{9}{11}$

(c) $\frac{5}{16} + 5\frac{1}{6}$ $\qquad$ **(d)** $1\frac{3}{7} + \frac{5}{3}$

2. Evaluate each of the following without using a calculator. Write each answer in its simplest form, where necessary.

(a) $2\frac{5}{6} - 2\frac{1}{6}$ $\qquad$ **(b)** $5\frac{3}{7} + \frac{19}{7}$

(c) $5\frac{5}{9} - \frac{19}{6}$ $\qquad$ **(d)** $2\frac{1}{6} - 1\frac{2}{5}$

3. Sara spent $\frac{1}{4}$ h preparing the ingredients for a dish. She then boiled some eggs for $\frac{1}{6}$ h, before putting the dish in the oven for $1\frac{2}{5}$ h. Find the total time Sara took in hours.

1. Given a pair of fractions, mixed numbers or improper fractions, how do I know if they are equivalent?
2. How did what I learnt about finding lowest common multiple help me in adding and subtracting unlike fractions and mixed numbers with different denominators?

Basic | Intermediate | Advanced

Exercise

Do not use a calculator for this exercise.

1. Express each fraction in its simplest form.

(a) $\frac{9}{81}$ (b) $\frac{10}{12}$

(c) $\frac{36}{90}$ (d) $\frac{56}{84}$

2. Compare the following pairs of fractions by filling the blanks with '>', '<' or '='.

(a) $\frac{2}{5}$ ____ $\frac{11}{20}$ (b) $\frac{3}{16}$ ____ $\frac{15}{80}$

(c) $\frac{2}{3}$ ____ $\frac{5}{8}$ (d) $\frac{13}{18}$ ____ $\frac{10}{12}$

3. Convert each mixed number to an improper fraction in its simplest form.

(a) $12\frac{2}{3}$ (b) $4\frac{3}{8}$

(c) $5\frac{4}{6}$ (d) $9\frac{72}{81}$

4. Convert each improper fraction to a mixed number. Write each answer in its simplest form.

(a) $\frac{35}{2}$ (b) $\frac{21}{14}$

(c) $\frac{50}{6}$ (d) $\frac{30}{18}$

5. Compare the following pairs of numbers using '>', '<' or '='.

(a) $1\frac{4}{12}$ ____ $1\frac{1}{3}$ (b) $1\frac{2}{3}$ ____ $1\frac{9}{12}$

(c) $2\frac{12}{54}$ ____ $2\frac{6}{27}$ (d) $\frac{70}{12}$ ____ $\frac{17}{3}$

(e) $\frac{11}{7}$ ____ $\frac{19}{12}$ (f) $\frac{11}{9}$ ____ $\frac{77}{63}$

(g) $1\frac{1}{2}$ ____ $\frac{12}{8}$ (h) $7\frac{4}{7}$ ____ $\frac{105}{14}$

6. Find the value of each of the following. Write each answer as a mixed number in its simplest form where necessary.

(a) $\frac{8}{17}+\frac{5}{17}$ (b) $\frac{13}{20}+\frac{11}{20}$

(c) $\frac{2}{3}+\frac{5}{9}$ (d) $\frac{4}{6}+\frac{4}{9}$

(e) $\frac{3}{4}+\frac{5}{6}$ (f) $\frac{5}{13}-\frac{2}{13}$

(g) $\frac{17}{20}-\frac{8}{20}$ (h) $3-\frac{2}{3}$

(i) $\frac{1}{2} \quad \frac{2}{5}$ (j) $\frac{2}{3}-\frac{1}{7}$

7. Find the value of each of the following. Write each answer as a mixed number in its simplest form where necessary.

(a) $1\frac{1}{7}+2\frac{5}{7}$ (b) $1\frac{2}{9}+\frac{8}{9}$

(c) $2\frac{8}{15}+5\frac{7}{15}$ (d) $5\frac{2}{5}+2\frac{3}{10}$

(e) $1\frac{4}{15}+3\frac{2}{5}$ (f) $1\frac{3}{10}+\frac{5}{6}$

(g) $\frac{13}{4}+1\frac{5}{6}$ (h) $3\frac{5}{7}+\frac{9}{11}$

Exercise 2A

8. Find the value of each of the following. Write each answer as a mixed number in its simplest form where necessary.

(a) $10\frac{3}{5} - 3\frac{1}{5}$ (b) $2\frac{3}{7} - \frac{4}{7}$

(c) $\frac{37}{18} - 1\frac{7}{18}$ (d) $2\frac{8}{11} - 2\frac{9}{22}$

(e) $10\frac{3}{8} - 6\frac{13}{24}$ (f) $\frac{15}{4} - 2\frac{1}{6}$

(g) $3\frac{5}{8} - 1\frac{5}{6}$ (h) $3\frac{5}{7} - \frac{7}{5}$

9. Convert the following list of fractions into their equivalent forms with the same denominator. Hence arrange them in ascending order.

$$\frac{1}{6}, \frac{4}{9}, \frac{2}{3}, \frac{11}{18}, \frac{5}{12}$$

10. For the following list of numbers,
(i) convert each number to an improper fraction in its simplest form where necessary,
(ii) arrange the numbers in ascending order.

$$3\frac{4}{5}, \frac{47}{15}, \frac{32}{10}, 3\frac{2}{3}, 3\frac{15}{25}$$

11. Find the value of each of the following. Write each answer as a mixed number in its simplest form where necessary.

(a) $\frac{3}{4} + \frac{3}{8} + \frac{5}{16}$ (b) $\frac{4}{6} - \frac{1}{9} - \frac{1}{3}$

(c) $\frac{3}{4} + \frac{5}{6} - \frac{1}{2}$ (d) $2\frac{1}{12} + 1\frac{1}{2} + 1\frac{5}{6}$

(e) $5\frac{2}{3} + 4\frac{6}{7} + \frac{37}{21}$ (f) $4\frac{1}{2} + \frac{2}{7} + \frac{12}{35}$

(g) $4\frac{5}{6} - 2\frac{1}{9} - 2\frac{1}{3}$ (h) $3\frac{5}{21} - \frac{3}{4} - 1\frac{5}{7}$

(i) $3\frac{3}{4} + \frac{3}{8} - 1\frac{5}{16}$ (j) $10\frac{5}{8} - 3\frac{5}{6} + 1\frac{1}{3}$

12. Bernard had 1 kg of sugar. He used $\frac{1}{4}$ kg of sugar to bake cookies, $\frac{3}{5}$ kg of sugar to bake some cakes and the remaining sugar to bake muffins. How much sugar did he use to bake the muffins?

13. Ken drank $\frac{7}{10}$ l of water. If Nadia drank $\frac{1}{4}$ l more water than Ken, how many litres of water did Ken and Nadia drink altogether?

14. David cycles $1\frac{2}{5}$ km from his house to the library. He then cycles from the library to Raju's house. If the distance between the library and Raju's house is $2\frac{1}{2}$ km more than the distance between David's house and the library, what is the total distance that David cycles?

15. Joyce buys 3 small bottles of orange juice and 2 large bottles of apple juice. The volume of apple juice in a large bottle is $1\frac{3}{8}$ l more than the volume of orange juice in each small bottle. If each small bottle contains $1\frac{1}{4}$ l of orange juice, what is the total volume of drinks that Joyce buys altogether?

16. Shaha divided some cakes equally into 6 boxes. She gave away 5 boxes and kept 1 box for herself. She ate $1\frac{1}{4}$ cakes in the box. If Shaha is left with $1\frac{1}{12}$ cakes, how many cakes did she pack into the 6 boxes altogether?

2.3 Multiplying fractions and mixed numbers

A. Multiplying fractions and mixed numbers by integers

In primary school, we learnt that products such as 2×6 means 2 groups of 6, which gives a value of 12.

2 groups of 6 gives 12

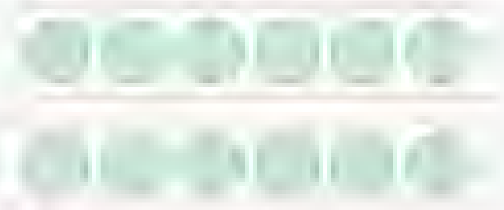


Fig. 2.3

How do we find the product of a fraction or a mixed number and an integer, e.g. $\frac{3}{5}$ of 20, or $1\frac{5}{6}$ of 3?

Worked Example 8

Multiplying fraction and mixed number by integer

Without using a calculator, evaluate

(a) $\frac{3}{5}$ of 20, **(b)** $5 \times \frac{3}{10}$, **(c)** $1\frac{5}{6} \times 3$.

Solution

(a) **Method 1:**

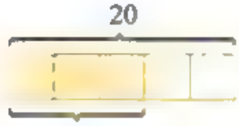
5 units = 20

1 unit = $20 \div 5$

= 4

3 units = 3×4

= 12



Method 2:

$$\frac{3}{5} \times 20$$
$$= \frac{3 \times 20}{5}$$
$$= \frac{60}{5}$$
$$= 12$$

Method 3:

$$\frac{3}{\cancel{5}_1} \times \cancel{20}^4 = 3 \times 4$$
$$= 12$$

Problem-solving Tip

In [illegible], we divide the denominator of the fraction, 5, and the integer, 20, by their highest common factor, 5.

(b) Method 1:

$$5 \times \frac{3}{10} = \frac{5 \times 3}{10}$$
$$= \frac{15}{10}$$
$$= 1\frac{5}{10}$$
$$= 1\frac{1}{2}$$

Method 2:

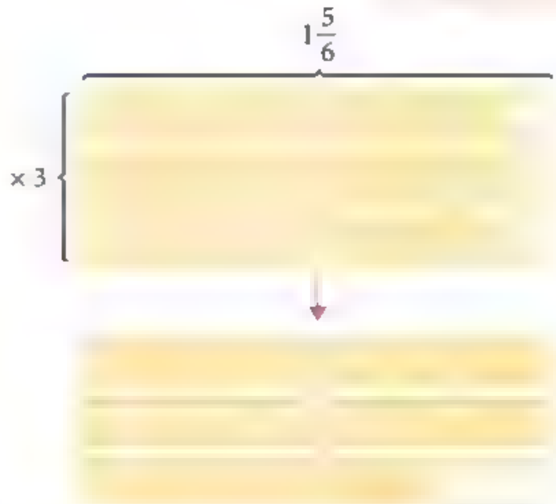
$$\cancel{5}^{1} \times \frac{3}{\cancel{10}_{2}} = \frac{1 \times 3}{2}$$
$$= \frac{3}{2}$$

Attention

What is the value of $\frac{3}{10} \times 5$?
In primary school, we learnt that changing the order of factors does not change the product, e.g. $2 \times 3 = 3 \times 2$. This is the same when multiplying fractions.

(c) Method 1:

$$1 \times 3 = 3$$
$$\frac{5}{\cancel{6}_{2}} \times \cancel{3}^{1} = \frac{5}{2}$$
$$\therefore 1\frac{5}{6} \times 3 = 3\frac{5}{2}$$
$$= 5\frac{1}{2}$$



Method 2:

$$1\frac{5}{6} \times 3 = \frac{11}{\cancel{6}_{2}} \times \cancel{3}^{1}$$
$$= \frac{11}{2}$$
$$= 5\frac{1}{2}$$

[illegible]

Exercise 2B

[illegible]

Without using a calculator, find the value of the following. Write each answer in its simplest form.

(a) $\frac{3}{7}$ of 28

(b) $\frac{1}{6} \times 81$

(c) $36 \times 2\frac{7}{8}$

Problem-solving Tip

In [illegible], the mixed number is converted to an [illegible] first for multiplication

B. Multiplying two fractions or mixed numbers

Multiplying two fractions

Part 1:

1. (a) Fold a piece of paper into 4 equal parts. Shade one part.
   $\frac{1}{4}$ of the paper is shaded as shown in Fig. 2.4(a).

   (b) Fold the paper again into $\frac{1}{2}$ vertically.
   Outline $\frac{1}{2}$ of the shaded part as shown in Fig. 2.4(b).
   The outlined portion shows $\frac{1}{2}$ of $\frac{1}{4}$ of the paper.
   What is the fraction of the paper that is outlined?

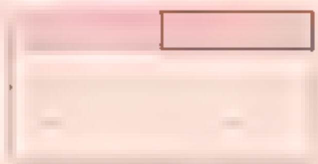


Fig. 2.4(b)

   (c) From part (b), we have $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{2} \times \frac{1}{4}$

$$= \quad$$

$$= \frac{1}{2 \times \quad}$$

2. (a) The rectangle in Fig. 2.5 is divided into 2 equal parts. Shade 1 part.
   What is the fraction of the paper that is shaded?

   (b) On Fig. 2.5, draw horizontal lines to divide the shaded portion into 4 equal parts.
   Outline $\frac{1}{4}$ of the shaded part. What is the fraction of the paper that is outlined?

   (c) From part (b), we have of $\frac{1}{2} = \quad \times \frac{1}{2}$

$$= \quad$$

$$= \frac{1}{\quad \times 2}$$

Part 2:

3. (a) Use the rectangle in Fig. 2.6 to determine the value of $\frac{3}{4} \times \frac{1}{2}$.
   $\frac{3}{4} \times \frac{1}{2} =$

   (b) Based on Questions 1(c) and 2(c), suggest how $\frac{3}{4} \times \frac{1}{2}$ can be evaluated without the aid of a figure.
   Try out what you have suggested. Does the suggested method give the same product as Question 3(a)?

We have seen that changing the order of two integers in a multiplication does not change the value of their product, i.e. $2 \times 3 = 3 \times 2$. From the Investigation on page 36, we see that this is the same when multiplying two fractions.

To multiply two fractions, we multiply the numerators and the denominators separately.

Worked Example 9

Multiplying two fractions or mixed numbers

Find the value of each of the following products. Write each answer in its simplest form.

(a) $\frac{1}{5} \times \frac{5}{12}$ (b) $\frac{2}{5} \times \frac{7}{4}$ (c) $1\frac{7}{15} \times 2\frac{3}{11}$

*Solution

(a) Method 1:

$$\frac{1}{5} \times \frac{5}{12} = \frac{1 \times 5}{5 \times 12}$$
$$= \frac{5}{60}$$
$$= \frac{1}{12}$$ reduce to the simplest form

Method 2:

$$\frac{1}{\cancel{5}_1} \times \frac{\cancel{5}^1}{12} = \frac{1}{12}$$

(b) $$\frac{2}{5} \times \frac{7}{4} = \frac{1 \times 7}{5 \times 2}$$
$$= \frac{7}{10}$$

(c) $$1\frac{7}{15} \times 2\frac{3}{11} = \frac{\cancel{22}^2}{\cancel{15}_3} \times \frac{\cancel{25}^5}{\cancel{11}_1}$$ convert to improper fractions
$$= \frac{10}{3}$$
$$= 3\frac{1}{3}$$ leave answer as a mixed number

Reflection

In part (a), how are the two methods similar to each other? Which method do you prefer? Why?

Practice Now 9

Find the value of each of the following products without using a calculator. Write each answer in its simplest form.

(a) $\frac{1}{5} \times \frac{2}{3}$ (b) $\frac{1}{3} \times \frac{3}{13}$

(c) $\frac{4}{15} \times \frac{35}{12}$ (d) $1\frac{7}{15} \times 2\frac{10}{21}$

Worked Example 10

Problem involving multiplication of fractions

$\frac{5}{8}$ of the audience at a musical are females and $\frac{1}{3}$ of the males are boys.

(i) What fraction of the audience are boys?

(ii) If there are 150 boys, how many people are there at the musical?

*Solution

(i) Fraction of the audience who are males

$= 1 - \frac{5}{8}$

$= \frac{3}{8}$

Fraction of the audience who are boys

$= \frac{1}{3} \times \frac{3}{8}$

$= \frac{1}{8}$

(ii) $\frac{1}{8}$ of the number of people in the audience = 150

$\therefore$ number of people in the audience $= 150 \times 8$

$= 1200$

Practise Now 10

Exercise 2B

Cheryl baked some muffins. $\frac{3}{5}$ of the muffins were chocolate and the rest were vanilla. She sold $\frac{1}{2}$ of the vanilla muffins and kept the rest.

(i) What fraction of the total number of muffins did Cheryl sell?

(ii) If Cheryl sold 138 muffins, how many chocolate muffins did she bake?

2.4 Dividing fractions and mixed numbers

A. Dividing fractions and mixed numbers by integers

Let us now learn to divide a fraction by an integer by looking at an example. What do we get when we divide $\frac{1}{4}$ by 2?

From Fig. 2.7, we see that this is the same as finding $\frac{1}{2}$ of $\frac{1}{4}$, i.e. $\frac{1}{4} \div 2 = \frac{1}{2} \times \frac{1}{4}$.

$\frac{1}{4}$ $\qquad$ $\frac{1}{4} \div 2 = \frac{1}{2} \times \frac{1}{4}$

Fig. 2.7

That is, dividing by an integer is equivalent to *multiplying by its reciprocal*.

Attention

The *reciprocal* of 2 is $\frac{1}{2}$.

Example 11

[illegible]

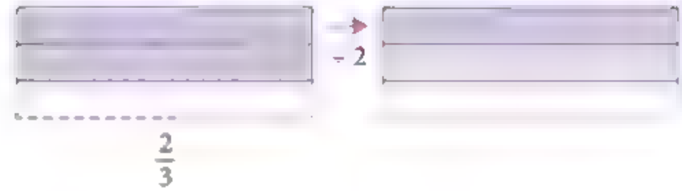
Find the value of each of the following. Write each answer in its simplest form.

(a) $\frac{2}{3} \div 2$ **(b)** $\frac{7}{4} \div 2$ **(c)** $6\frac{3}{5} \div 3$

Solution

(a) Method 1:

$$\frac{2}{3} \div 2 = \frac{1}{3}$$

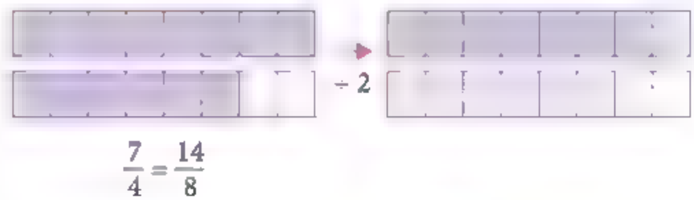


Method 2:

$$\begin{aligned} \frac{2}{3} \div 2 &= \frac{1}{2} \text{ of } \frac{2}{3} \\ &= \frac{1}{\cancel{2}_1} \times \frac{\cancel{2}^1}{3} \\ &= \frac{1}{3} \end{aligned}$$

(b) Method 1:

$$\begin{aligned} \frac{7}{4} \div 2 &= \frac{14}{8} \div 2 \\ &= \frac{7}{8} \end{aligned}$$



Method 2:

$$\begin{aligned} \frac{7}{4} \div 2 &= \frac{7}{4} \times \frac{1}{2} \\ &= \frac{7}{8} \end{aligned}$$

(c)

$$\begin{aligned} 6\frac{3}{5} \div 3 &= \frac{\cancel{33}^{11}}{5} \times \frac{1}{\cancel{3}_1} \\ &= \frac{11}{5} \\ &= 2\frac{1}{5} \end{aligned}$$

Reflection

In Method 1 of part (b), why do we convert $\frac{7}{4}$ to $\frac{14}{8}$?

Problem-solving Tip

For division, always convert mixed numbers to [illegible]

Practice Now

Find the value of each of the following without using a calculator. Write each answer in its simplest form.

(a) $\frac{1}{4} \div 3$ **(b)** $\frac{11}{12} \div 11$

(c) $\frac{12}{7} \div 8$ **(d)** $2\frac{3}{4} \div 9$

B. Dividing by fractions and mixed numbers

In Section 2.4A, we have learnt that dividing by an integer is the same as multiplying by its reciprocal.

Fig. 2.8

This also applies when dividing by a fraction. For example, when we divide a pie into $\frac{1}{4}$-pieces as shown in Fig. 2.8, we get 4 pieces altogether, i.e.

$$\begin{aligned} 1 \div \frac{1}{4} &= 1 \times 4 \\ &= 4 \end{aligned}$$

Fig. 2.9

What happens when we divide $\frac{3}{4}$ of the pie into $\frac{1}{4}$-pieces? As shown in Fig. 2.9, we get 3 pieces.

$$\begin{aligned} \frac{3}{4} \div \frac{1}{4} &= \frac{3}{4} \times 4 \\ &= 3 \end{aligned}$$

Example 12

Dividing by fraction and mixed number

Evaluate each of the following without using a calculator.

(a) $4 \div \frac{2}{3}$ (b) $\frac{4}{9} \div \frac{6}{11}$

(c) $\frac{8}{5} \div \frac{7}{3}$ (d) $2\frac{2}{9} \div 1\frac{2}{3}$

Solution

(a) $$\begin{aligned} 4 \div \frac{2}{3} &= 4 \times \frac{3}{2} \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

(b) $$\begin{aligned} \frac{4}{9} \div \frac{6}{11} &= \frac{\cancel{4}^{2}}{9} \times \frac{11}{\cancel{6}_{3}} \\ &= \frac{2}{9} \times \frac{11}{3} \\ &= \frac{22}{27} \end{aligned}$$

(c) $$\begin{aligned} \frac{8}{5} \div \frac{7}{3} &= \frac{8}{5} \times \frac{3}{7} \\ &= \frac{24}{35} \end{aligned}$$

(d) $2\frac{2}{9} \div 1\frac{2}{3} = \frac{20}{9} \div \frac{5}{3}$

$= \frac{\cancel{20}}{9} \times \frac{3}{\cancel{5}}$

$= \frac{4}{3}$

$= 1\frac{1}{3}$

dividing by a fraction is equivalent to multiplying by its reciprocal

Similar and Further Questions
Exercise 2B

1. Without using a calculator, find the value of each of the following.

(a) $9 \div \frac{1}{6}$ (b) $\frac{7}{8} \div \frac{21}{32}$

(c) $\frac{12}{5} \div \frac{7}{3}$ (d) $\frac{9}{4} \div \frac{15}{16}$

(e) $2\frac{7}{10} \div \frac{6}{25}$ (f) $4\frac{4}{5} \div 1\frac{1}{15}$

2. Li Ting and her brother were training for a competition. Li Ting ran $1\frac{3}{5}$ km while her brother ran $2\frac{11}{20}$ km more than her. If one lap of the running track is $\frac{2}{5}$ km long, determine

(i) the number of laps Li Ting's brother ran,

(ii) the number of laps Li Ting and her brother ran altogether.

Introductory Problem Revisited

Now that you have learnt about the basic operations involving fractions and mixed numbers, solve the problem and discuss your solution with your classmates.

1. How did what I know about finding common factors allow me to simplify multiplication and division involving fractions and mixed numbers?
2. What have I learnt in this section or chapter that I am still unclear of?

Basic | Intermediate

Exercise 2B

Do not use a calculator for this exercise.

1. Evaluate the following.

(a) $\frac{1}{6}$ of 96 (b) $\frac{5}{3}$ of 63

(c) $\frac{9}{4} \times 42$ (d) $21 \times 2\frac{1}{3}$

(e) $\frac{1}{2} \times \frac{1}{5}$ (f) $\frac{3}{10} \times \frac{11}{12}$

(g) $\frac{5}{9} \times \frac{15}{16}$ (h) $\frac{15}{22} \times \frac{4}{9}$

(i) $2\frac{3}{5} \times \frac{15}{26}$ (j) $\frac{15}{8} \times \frac{4}{3}$

(k) $\frac{7}{11} \times 1\frac{4}{7}$ (l) $1\frac{1}{9} \times 2\frac{5}{8}$

2. Evaluate the following.

(a) $\frac{1}{2} \div 3$ (b) $\frac{16}{21} \div 8$

(c) $\frac{24}{5} \div 2$ (d) $5\frac{25}{26} \div 10$

(e) $6 \div \frac{3}{5}$ (f) $45 \div \frac{18}{11}$

(g) $\frac{3}{4} \div \frac{9}{10}$ (h) $\frac{8}{15} \div \frac{32}{25}$

(i) $\frac{15}{4} \div \frac{5}{2}$ (j) $1\frac{7}{9} \div \frac{4}{3}$

(k) $9\frac{2}{9} \div 9$ (l) $2\frac{2}{15} \div 1\frac{23}{25}$

3. Sara spent a total of 8 hours volunteering last year. Given that her visit to a care home made up $\frac{4}{7}$ of the total time on volunteering, find the amount of time she spent visiting the care home.

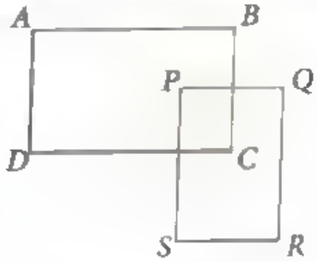
4. At a concert, $\frac{3}{4}$ of the audience are adults and $\frac{2}{5}$ of the children are girls.
(i) What fraction of the audience are girls?
(ii) If there are 90 girls, how many boys are there in the audience?

5. Cheryl cut $\frac{9}{10}$ kg of butter into 12 slices of equal mass. She then used $2\frac{1}{4}$ slices of butter to bake cupcakes. What is the total mass of butter Cheryl used for the cupcakes?

6. Li Ting, Waseem, Yasir and Nadia shared a gift for their friend. Li Ting paid $\frac{1}{4}$ of the cost of the gift. Waseem paid $\frac{1}{6}$ of the cost of the gift. If Yasir paid $\frac{3}{7}$ of the remaining cost of the gift and Nadia paid $16, how much did the gift cost?

7. $\frac{3}{4}$ of the spectators at a football match are male. $\frac{1}{9}$ of the males are boys and $\frac{1}{6}$ of the females are girls. If there are 56 more boys than girls, how many spectators are there at the football match altogether?

8. A figure is made up of two rectangles *ABCD* and *PQRS*. $\frac{1}{8}$ of *ABCD* overlaps with $\frac{1}{5}$ of *PQRS*. If the area of *PQRS* is 13 cm², find the area of the figure.



9. Albert spent $\frac{1}{4}$ and $\frac{2}{5}$ of his salary on transport and food respectively. He saved $\frac{5}{7}$ of the remainder and donated the rest equally among 4 charities. What is Albert's salary if each charity receives PKR 1400?

10. Joyce has 10 *l* of lemonade and 12 glasses. She keeps $\frac{2}{5}$ of the lemonade and pours the remaining into glasses. If she pours exactly $\frac{3}{10}$ *l* of lemonade into each glass, how many more glasses does she need?

Exercise 2B

After spending $\frac{1}{3}$ of his pocket money on food and $\frac{1}{7}$ of the remaining amount on a pen, Imran had PKR 240 left. How much pocket money did Imran have at first?

In this chapter, we learnt about a type of number called fractions. Fractions are commonly used to express quantities that are not whole numbers, allowing for more precise and flexible representations in various contexts. They are also useful in the division of quantities: for instance, if 4 bags of rice are distributed equally to 7 people, we can say each person receives $\frac{4}{7}$ of the rice.

Besides mathematics, understanding fractions is crucial in other fields like science and engineering. They are also widely used in our everyday life, such as measurements, calculations and comparisons. It is thus essential to develop a solid grasp of concepts in fractions to build a strong foundation in mathematics and problem solving. In this chapter, we built upon what we have learnt in the previous chapter, such as finding the lowest common multiple and the highest common factor of two numbers. We have also extended what we know about adding, subtracting, multiplying and dividing whole numbers to operations involving fractions. How do we apply what we have learnt about fractions to other numbers, such as decimals?

1. **Equivalent fractions** are fractions that have the same value, e.g. $\frac{1}{2}$ and $\frac{4}{8}$, $\frac{9}{12}$ and $\frac{3}{4}$.

- simplifying a fraction
  - Divide both the numerator and denominator by their common factor.
- comparing two or more fractions of different denominators (**unlike fractions**)
- performing operations such as addition and subtraction
  - Find the LCM of the denominators and express the fractions in their equivalent forms, with the same denominator (**like fractions**).

2. **Mixed numbers** and **improper fractions** are greater than 1. They can be converted from one form to the other. They can be simplified and expressed in equivalent forms using the same techniques for proper fractions.

Example:

$$2\frac{3}{4} = \frac{4}{4} + \frac{4}{4} + \frac{3}{4}$$
$$= \frac{11}{4}$$

$$\frac{8}{3} = \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$$
$$= 2\frac{2}{3}$$

3. **Adding** and **subtracting** fractions and mixed numbers:

(a) When adding or subtracting *like fractions*, add or subtract the numerators.

(b) When adding or subtracting *unlike fractions*, convert the fractions to like fractions first.

Examples:

$$\frac{3}{8}+\frac{7}{8}=\frac{10}{8}$$
$$=1\frac{2}{8}$$
$$=1\frac{1}{4}$$

$$\frac{5}{6}-\frac{3}{4}=\frac{10}{12}-\frac{9}{12}$$
$$=\frac{1}{12}$$

(c) When adding *mixed numbers*, separately *add*

- the whole numbers, and
- the fractions,

before *adding* the fraction to the whole number to obtain the result.

Examples:

$$1\frac{3}{4}+2\frac{2}{3}=1\frac{9}{12}+2\frac{8}{12}$$
$$\frac{9}{12}+\frac{8}{12}=\frac{17}{12} \text{ and } 1+2=3$$
$$\therefore 1\frac{3}{4}+2\frac{2}{3}=3\frac{17}{12}$$
$$=4\frac{5}{12}$$

(d) When subtracting *mixed numbers*, separately *subtract*

- the whole numbers, and
- the fractions,

before *adding* the fraction to the whole number to obtain the result.

Example:

$$4\frac{5}{8}-1\frac{1}{6}=4\frac{15}{24}-1\frac{4}{24}$$
$$\frac{15}{24}-\frac{4}{24}=\frac{11}{24} \text{ and } 4-1=3$$
$$\therefore 4\frac{5}{8}-1\frac{1}{6}=3\frac{11}{24}$$

4. **Multiplying and dividing** fractions and mixed numbers:

(a) To multiply a fraction by an integer, we divide the integer by the denominator of the fraction and multiply the result by its numerator, or vice versa.

Example:

$$\frac{3}{\cancel{8}} \times \cancel{20} = 3 \times 4$$
$$= 12$$

(b) To multiply two fractions, we multiply the numerators and the denominators separately.

Example:

$$\frac{3}{8} \times \frac{1}{4} = \frac{3 \times 1}{8 \times 4}$$
$$= \frac{3}{32}$$

(c) Dividing by an integer or a fraction is equivalent to multiplying by the reciprocal of the integer or the fraction.

Examples:

$$\frac{6}{7} \div 3 = \frac{\cancel{6}}{7} \times \frac{1}{\cancel{3}}$$
$$= \frac{2}{7}$$

$$\frac{8}{3} \div \frac{16}{9} = \frac{\cancel{8}}{3} \times \frac{9}{\cancel{16}}$$
$$= \frac{3}{2}$$
$$= 1\frac{1}{2}$$

(d) When a multiplication or division involves a mixed number, express it as an improper fraction first.

Example:

$$3\frac{2}{15} \div 2\frac{1}{9} = \frac{47}{15} \div \frac{19}{9}$$
$$= \frac{47}{\cancel{15}} \times \frac{\overset{3}{\cancel{9}}}{19}$$
$$= \frac{141}{95}$$
$$= 1\frac{46}{95}$$

CHAPTER 3

Decimals

The British pound sterling underwent decimalisation on 15 February 1971, known as the Decimal Day. Prior to this, each British pound sterling was subdivided into 20 shillings, with each shilling being equivalent in value to 12 pence. With decimalisation the pound retained its value, but the penny was revalued such that 1 pound was subdivided into 100 pence. That is, 1 penny = 0.01 pound.

Today, most countries use decimal currencies, with one basic currency unit (e.g. the dollar) that is commonly divided into 100 subunits (e.g. cents). The term 'decimal' comes from the Latin word '*decem*', meaning ten. Besides money, we use decimals every day in situations dealing with length, volume, etc.

Can you think of other real-life examples involving the use of decimals?

Learning Outcomes

What will we learn in this chapter?

- What terminating and recurring decimals are
- How to express a fraction as a decimal, and vice versa
- How to perform operations on decimals
- What metric conversions are

Introductory Problem

1. Convert each of the following fractions to a decimal, leaving your answer to 4 decimal places if the answer is not exact. For part **(e)**, leave your answer to 10 decimal places.

(a) $\frac{1}{2}$ **(b)** $\frac{3}{8}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{6}$ **(e)** $\frac{1}{7}$

2. Convert each of the following decimals to a fraction, leaving your answer in its simplest form.

(a) 0.3 **(b)** 0.4 **(c)** 0.25 **(d)** 0.167 **(e)** 0.625

In this chapter, we are going to learn more about the relationship between fractions and decimals, and to perform basic operations involving decimals.

3.1 Decimals and fractions

A. Fractions and terminating decimals

In our everyday life, we encounter decimals, such as

0.73

The digit 0 is in the *ones* place.

decimal point

The digit 7 is in the *tenths* place.

The digit 3 is in the *hundredths* place.

In fractional form,

$$0.7\underline{3} = \frac{73}{100}$$

Check the place value of the last digit of the decimal. Here, the last digit is in the *hundredths* place.

0.73 is thus 73 hundredths. This is written as $\frac{73}{100}$. This fraction is in its simplest form.

In decimals with three digits after the decimal point, such as 0.356, the digit 6 occupies the *thousandths* place. That is, the value of the digit 6 is 6 thousandths.

To express 0.356 as a fraction,

$$0.35\underline{6} = \frac{356}{1000} = \frac{89}{250}$$

Step 1:
Check the place value of the last digit of the decimal. Here, the last digit is in the *thousandths* place.

Step 2:
0.356 is read as 356 thousandths. This is written as $\frac{356}{1000}$.

Step 3:
Reduce $\frac{356}{1000}$ to its simplest form by dividing both numerator and denominator by the common factor 4.

We may also express fractions as decimals. What if the denominator of the fraction is not 10, 100 or 1000?

There are two ways in which a fraction such as $\frac{1}{2}$ or $\frac{5}{8}$ can be converted to their decimal representations.

Example 1

Converting fractions to decimals

Express each of the following fractions as a decimal

(a) $\frac{1}{2}$ (b) $1\frac{5}{8}$

Solution

(a) We convert the denominator to 10 to find the number of tenths.

$$\frac{1}{2} = \frac{5}{10}$$
$$= 0.5$$

(b) In $1\frac{5}{8}$, the digit in the ones place is 1. We determine the digits after the decimal point by looking at $\frac{5}{8}$.

Method 1: Converting the denominator to 1000

$$\begin{aligned}\frac{5}{8} &= \frac{5 \times 125}{8 \times 125} \\ &= \frac{625}{1000} \\ &= 0.625\end{aligned}$$

$$\therefore 1\frac{5}{8} = 1.625$$

Method 2: Using long division

Step 1:
Since 5 is less than 8, the first digit of the quotient is 0.

Step 2:
Place a decimal point and a zero after 5 to get 5.0. Add a decimal point after 0 in the quotient.

$$\begin{array}{r} 0.625 \\ 8 \overline{)\,5.000} \\ -\,4\,8 \\ \hline 2\,0 \\ -\,1\,6 \\ \hline 4\,0 \\ -\,4\,0 \\ \hline 0 \\ \hline \end{array}$$

Step 3:
$50 \div 8 = 6$ remainder 2.
6 is written in the quotient while 48 is written below 5.0.

Step 4:
The remainder 2 is less than 8.
Add a zero after 5.0 to get 5.00 and bring it down. Repeat steps 3 and 4 until the remainder = 0.

$$\therefore 1\frac{5}{8} = 1.625$$

1. Express each of the following as a fraction or a mixed number in its simplest form.
   (a) 0.05 (b) 0.13
   (c) 2.04 (d) 7.354

2. Express each of the following as a decimal.
   (a) $\frac{2}{5}$ (b) $\frac{9}{20}$ (c) $\frac{78}{250}$
   (d) $1\frac{1}{20}$ (e) $2\frac{3}{4}$

3. Compare the following pairs of numbers using '>', '=' or '<'.
   (a) 0.076 ☐ 0.0706 (b) 4.050 ☐ 4.05
   (c) $2\frac{33}{1000}$ ☐ 2.330 (d) 90.37 ☐ $\frac{723}{8}$

Introductory Problem Revisited

Now that you have learnt how to express a fraction as a decimal and vice versa, solve the .
When expressing the fractions in Question 1(**c**) to (**e**) as decimals, are you able to use as shown in Worked Example 1(**b**)? What is the difference between the decimals obtained in Question 1(**c**) to (**e**), and those obtained in Question 1(**a**) and (**b**)?

B. Fractions and recurring decimals

In Worked Example 1 of Section 3.1A, 0.5 and 0.625 each have a fixed number of digits after the decimal point. Such decimals are called **terminating decimals** because the digits end (or terminate).

Which decimals in the Introductory Problem are terminating?

On the other hand, for fractions such as $\frac{1}{3}$ and $\frac{1}{6}$, their equivalent decimals are 0.3333… and 0.1666….

These decimals, which contain digits that *repeat indefinitely*, are called **recurring** (or **repeating**) decimals.

When writing recurring decimals, we place a dot above the repeating digit, i.e.

$\frac{1}{3} = 0.3333\ldots = 0.\dot{3}$, and

$\frac{1}{6} = 0.1666\ldots =$ ☐.

For recurring decimals that occur in a pattern, a dot is placed over the first and last digits of the recurring set of numbers.

$$\frac{1}{7} = 0.\underbrace{142\,857}_{\text{set 1}}\ \underbrace{142\,857}_{\text{set 2}}\ldots = 0.\dot{1}42\,85\dot{7}$$

For recurring decimals such as 0.333… and 3.142 857 142 857 142 857…, we use a dot or dots to represent which digit or digits repeat in a and manner.

Attention

Sometimes, a bar (called a *vinculum*) is used in place of a dot, e.g. $\frac{1}{3} = 0.\overline{3}$.

Attention

In the bar notation, a single bar is placed over all the repeating digits:
$\frac{1}{7} = 0.\overline{142\,857}$.

Exercise 3A

1. Express the following recurring decimals using dot notation.
   (a) 4.4444 ... (b) 15.310 310 310...
   (c) 20.164 646 4...

2. Write the following decimals to 6 decimal places.
   (a) $0.\dot{4}\dot{7}$ (b) $0.0\dot{2}\dot{3}$
   (c) $1.\dot{2}0\dot{3}$

The method of finding an equivalent fraction with a denominator of 10, 100, 1000, . . . , as shown in Worked Example 1, cannot be applied to convert fractions such as $\frac{1}{3}$ to a decimal. Why?
Instead, we will use long division to determine the equivalent decimals for these fractions.

Worked Example 2

Converting fraction to recurring decimal

Convert $\frac{2}{3}$ to a recurring decimal.

Solution

$$\begin{array}{r} 0.666 \\ 3\overline{)\,2.000} \\ -1\,8 \\ \hline 20 \\ -\,18 \\ \hline 20 \\ -\,18 \\ \hline 2 \end{array}$$

$\therefore \frac{2}{3} = 0.\dot{6}$

Problem-solving Tip

In the working, we stopped the division after obtaining three repeating digits.
However, the long division can be truncated when we see that the remainder is a repeat of the divisor:

$$\begin{array}{r} 0.6 \\ 3\overline{)\,2.0} \\ -1\,8 \\ \hline 2 \end{array}$$

We start with the divisor 2

Once we obtain 2 as the remainder, we know that the digit 6 will repeat

Exercise 3A

Write each of the following fraction as a recurring decimal.

(a) $\frac{2}{9}$ (b) $\frac{5}{6}$

(c) $\frac{1}{11}$ (d) $\frac{3}{7}$

Converting recurring decimal to fraction

1. Express $0.\dot{4}$ as a fraction using the following steps. Let $x = 0.\dot{4}$.
   (a) Multiply x by 10:
   $10x = 4.\dot{4}$
   (b) Subtract x from $10x$:
   $10x - x = \_\_\_\_ - 0.\dot{4}$
   (c) The left-hand side of the equation in part **(b)** is $9x$. Thus, $9x = \_\_\_\_$
   $x = \frac{\_\_\_\_}{9}$

Attention

$10x$, which means 'x multiplied by 10', is an *algebraic expression*. When x represents a number, $10x$ means that the number is multiplied by 10. We will learn more about algebraic expressions in Chapter 6

2. Copy and complete Table 3.1.

	x	$10x$	$10x - x$	Determine the fractional form of x
(a)	$0.\dot{8}$	$8.\dot{8}$	$\_\_ - 0.\dot{8} = \_\_$	$10x - x = \_\_$ $9x = \_\_$ $x = \_\_$
(b)	$1.\dot{5}$		$\_\_ - \_\_ = \_\_$	

Table 3.1

3. Explain why the following steps are carried out in Questions 1 and 2
   (i) The recurring decimal x is multiplied by 10.
   (ii) The recurring decimal x is subtracted from the recurring decimal $10x$.
4. (i) Explain why the steps in Question 1 cannot be used to convert $0.\dot{2}\dot{7}$ to a fraction.
   (ii) Suggest the steps to convert $0.\dot{2}\dot{7}$ to a fraction.
   (iii) Using the steps in part (ii), show that $0.\dot{2}\dot{7} = \frac{27}{99}$.

From the above Investigation, we see that an important step in converting a recurring decimal to a fraction is to *eliminate* the recurring digits after the decimal point. The *power of ten* that is multiplied to the decimal is selected with this aim. This is demonstrated in Worked Examples 3 and 4.

Attention

Powers of ten are numbers that are obtained by multiplying 10 by itself. The square of 10 ($10^2 = 100$) and the cube of 10 ($10^3 = 1000$) are examples of powers of ten. Can you give another example of a power of ten?

Worked Example 3

Converting recurring decimal to fraction

Write $0.\dot{2}2\dot{5}$ as a fraction in its simplest form.

Let $x = 0.\dot{2}2\dot{5}$

$$\therefore\ 1000x - x = 225.\dot{2}2\dot{5} - 0.\dot{2}2\dot{5}$$

$$999x = 225$$

$$x = \frac{225}{999}$$

$$= \frac{25}{111} \quad \text{reduce to the simplest form}$$

Problem-solving Tip

When a recurring decimal contains only recurring digits after the decimal point, the power of ten is selected based on the number of digits in each recurring set of numbers.

In Worked Example 3, the recurring set in x contains *three* digits. Hence, we multiply x by 10^3, i.e. $10^3 \times x = 1000x$, so that $1000x$ and x contain the same recurring sequence of digits, and $1000x - x$ eliminates the recurring digits.

Practice Now 3

Write each of the following recurring decimals as a fraction in its simplest form.

(a) $0.\dot{7}$
(b) $0.\dot{3}\dot{6}$
(c) $0.\dot{0}\dot{5}$
(d) $0.\dot{1}6\dot{7}$

Worked Example 4

Converting recurring decimal with some non-recurring digits to fractio

Write each of the following decimals as a fraction in its simplest form.

(a) $0.7\dot{3}$

(b) $0.7\dot{3}\dot{1}$

(c) $0.73\dot{1}$

Solution

(a) Let $x = 0.7\dot{3}$

$$\begin{aligned} 10x &= 7.\dot{3} \\ 100x &= 73.\dot{3} \\ 100x - 10x &= 73.\dot{3} - 7.\dot{3} \\ 90x &= 66 \\ x &= \frac{66}{90} \\ &= \frac{11}{15} \quad \text{reduce to the simplest form} \end{aligned}$$

(b) Let $x = 0.7\dot{3}\dot{1}$

$$\begin{aligned} 10x &= 7.\dot{3}\dot{1} \\ 1000x &= 731.\dot{3}\dot{1} \\ 1000x - 10x &= 731.\dot{3}\dot{1} - 7.\dot{3}\dot{1} \\ 990x &= 724 \\ x &= \frac{724}{990} \\ &= \frac{362}{495} \quad \text{reduce to the simplest form} \end{aligned}$$

(c) Let $x = 0.73\dot{1}$

$$\begin{aligned} 100x &= 73.\dot{1} \\ 1000x &= 731.\dot{1} \\ 1000x - 100x &= 731.\dot{1} - 73.\dot{1} \\ 900x &= 658 \\ x &= \frac{658}{900} \\ &= \frac{329}{450} \quad \text{reduce to the simplest form} \end{aligned}$$

Problem-solving Tip

When a recurring decimal contains non-recurring digit(s) after the decimal point, multiply the decimal by a power of ten so that the product contains only recurring digits after the decimal point.

The original decimal is multiplied again by another power of ten to obtain a second product that contains an identical sequence of recurring digits after the decimal point.

Reflection

Does it matter if the value of $1000x$ is evaluated instead of $100x$ in part (a)? Why or why not?

Similar and Further Questions
Exercise 3A

Write each of the following recurring decimals as a fraction in its simplest form.

(a) $0.8\dot{3}$

(b) $0.1\dot{3}\dot{6}$

(c) $0.41\dot{6}$

(d) $0.\dot{6}3\dot{0}$

Worked Example 5

Converting recurring decimal greater than 1 to fraction

Express $1.7\dot{5}$ as a mixed number.

Solution

Method 1:

$0.7\dot{5} = 1 + 0.7\dot{5}$

$$0.7\dot{5} = \frac{75-7}{90}$$
$$= \frac{68}{90}$$
$$= \frac{34}{45} \quad \text{reduce to the simplest form}$$
$$\therefore 1.7\dot{5} = 1\frac{34}{45}$$

Method 2:

Let $x = 1.7\dot{5}$

$$10x = 17.\dot{5}$$
$$100x = 175.\dot{5}$$
$$100x - 10x = 175.\dot{5} - 17.\dot{5}$$
$$90x = 158$$
$$x = \frac{158}{90}$$
$$= 1\frac{68}{90}$$
$$= 1\frac{34}{45}$$

Reflection

Which method do you prefer? Why?

Write each of the following recurring decimals as a fraction in its simplest form

(a) $1.\dot{5}\dot{6}$

(b) $2.6\dot{7}$

(c) $2.4\dot{3}\dot{2}$

(d) $3.\dot{2}48\dot{4}$

Identifying fractions with equivalent terminating decimals

$$\frac{1}{5}, \frac{1}{2}, \frac{2}{3}, \frac{5}{8}, \frac{2}{7}, \frac{7}{12}, \frac{11}{15}, \frac{13}{50}$$

1. Convert each of the fractions above to a decimal. Which fractions have equivalent decimals that are terminating?
2. Express the denominator of each fraction above as a product of its prime factors.
3. How do the denominators of fractions with equivalent terminating decimals differ from those with equivalent recurring decimals?
4. Discuss how you can identify a fraction with an equivalent decimal that is terminating.

1. How did what I learnt about fractions in Chapter 2 help me in expressing fractions as decimal representations and vice versa?
2. When converting a recurring decimal to a fraction, how do I identify the power of 10 that should be multiplied to eliminate the recurring digits?

Intermediate Basic

Exercise 3A

Do not use a calculator for this exercise.

1. Express each of the following as a fraction or a mixed number. Leave your answers in its simplest form.
   (a) 0.5 (b) 0.47
   (c) 0.36 (d) 0.05
   (e) 2.04 (f) 0.785
   (g) 7.355 (h) 1.040

2. Express each of the following as a decimal.
   (a) $\frac{3}{5}$ (b) $\frac{3}{20}$
   (c) $\frac{6}{20}$ (d) $\frac{23}{25}$
   (e) $3\frac{1}{2}$ (f) $5\frac{3}{4}$
   (g) $\frac{25}{4}$ (h) $5\frac{5}{8}$

3. Compare each pair of numbers using '>', '=' or '<'.
   (a) 1.15 □ 1.14 (b) 1.15 □ 1.2
   (c) 2.05 □ 2.050 (d) 0.21 □ 0.31
   (e) 3.1 □ 3.099 (f) 6.3 □ $6\frac{3}{10}$
   (g) $\frac{11}{5}$ □ 2.5 (h) 11.8705 □ $\frac{95}{8}$

4. Express the following recurring decimals using dot notation.
   (a) 0.2222… (b) 0.123 123 123…
   (c) 0.1777… (d) 0.234 242 424…

5. Write the following decimals to 6 decimal places.
   (a) $0.\dot{1}\dot{2}$ (b) $0.7\dot{1}$
   (c) $0.\dot{3}5\dot{3}$ (d) $0.33\dot{2}\,2\dot{4}$

6. Express each of the following fraction as a recurring decimal.
   (a) $\frac{7}{9}$ (b) $\frac{1}{18}$
   (c) $\frac{11}{15}$ (d) $\frac{5}{12}$

7. Express each of the following decimals as a fraction in its simplest form.
   (a) $0.\dot{5}$ (b) $0.\dot{6}$
   (c) $0.\dot{8}\dot{7}$ (d) $0.\dot{1}2\dot{1}$
   (e) $0.1\dot{2}$ (f) $0.0\dot{7}$
   (g) $0.7\dot{7}\dot{4}$ (h) $0.77\dot{4}$

8. Write each of the following decimals as a mixed number. Leave the answer in its simplest form.
   (a) $1.\dot{6}$ (b) $2.\dot{0}\dot{7}$
   (c) $3.\dot{4}1\dot{5}$ (d) $1.5\dot{7}$

9. Express the following recurring decimals using dot notation.
   (a) 1.017 317 3173… (b) 4.4444…
   (c) 20.022 323 23… (d) 7.147 575 75…

>

Basic | Intermediate

Exercise 3A

10. Write the following decimals to 9 decimal places.

(a) $0.0\dot{1}0\dot{2}$ (b) $12.1\dot{1}1\dot{2}$

(c) $10.00\dot{3}$ (d) $3.\dot{0}33\dot{2}$

11. Express each of the following fractions as a recurring decimal.

(a) $\frac{2}{11}$ (b) $\frac{3}{22}$

(c) $\frac{4}{7}$ (d) $\frac{1}{13}$

12. Express each of the following decimals as a fraction in its simplest form.

(a) $0.\dot{1}\dot{2}$ (b) $0.\dot{7}\dot{2}$

(c) $0.\dot{1}6\dot{2}$ (d) $0.\dot{0}79\dot{2}$

(e) $0.1\dot{5}$ (f) $0.00\dot{2}$

(g) $0.0\dot{0}\dot{2}$ (h) $0.16\dot{3}$

13. Express each of the following decimals as an improper fraction in its simplest form.

(a) $1.0\dot{7}$ (b) $2.11\dot{5}$

(c) $2.5\dot{0}\dot{7}$

14. Write each of the following fraction as a recurring decimal.
Hint: Perform long division until recurring digits are observed.

(a) $\frac{1}{17}$ (b) $\frac{1}{19}$

15. Express each of the following recurring decimals as a fraction in its simplest form.

(a) $0.\dot{3}8461\dot{5}$ (b) $0.\dot{4}2857\dot{1}$

(c) $0.18\dot{9}\dot{7}$ (d) $0.1\dot{8}9\dot{7}$

16. Express each of the following decimals as an improper fraction in its simplest form.

(a) $1.10\dot{8}\dot{7}$ (b) $2.2\dot{4}4\dot{6}$

17. Find the exact value of $3 - 2.\dot{9}$.

3.2 Operations involving decimals

A. Adding and subtracting decimals

In primary school, we have learnt how to add and subtract decimals with the same number of decimal places by aligning the decimal points.

Let us now learn how to perform these operations on decimals with different numbers of decimal places.

align decimal point

$$\begin{array}{r} 2.60 \\ +\ 6.78 \\ \hline 9.38 \end{array}$$

insert a zero in the hundredths place as a place holder

Inserting a zero after the last digit of a decimal does not change its value.

Exercise 3B

1. Find the sum of each pair of numbers.
   (a) 2.63 and 8.5
   (b) 5.7 and 6.24
   (c) 28.09 and 3.654
   (d) 14.15 and 21.96

2. Find the difference between each pair of numbers.
   (a) 8.63 and 6.59
   (b) 7 and 5.4
   (c) 38.9 and 3.65
   (d) 30 and 5.84

B. Multiplying decimals

We have learnt how to multiply fractions in Chapter 2. Let us apply what we have learnt to multiply decimals

Worked Example 8

Multiplying decimal with whole number

Without the use of a calculator, find each of the following products.

(a) 2.6×3 (b) 3.36×12

Solution

(a)

$$\begin{array}{r} 2.6 \\ \times \quad 3 \\ \hline 7.8 \\ \hline \end{array}$$

2.6 ← 1 decimal place (1 d.p.)

7.8 ← final answer has 1 d.p.

$\therefore 2.6 \times 3 = 7.8$

(b)

$$\begin{array}{r} 3.36 \\ \times \quad 12 \\ \hline 672 \\ 3360 \\ \hline 40.32 \\ \hline \end{array}$$

3.36 ← 2 decimal places (2 d.p.)

40.32 ← final answer has 2 d.p.

$\therefore 3.36 \times 12 = 40.32$

Attention

When a multiplication involves only one decimal, the final answer has the same number of decimal places as this decimal. Why is this so?

Hint: See Method 1 of Worked Example 7.

Exercise 3B

Without the use of a calculator, find each of the following products.

(a) 1.5×7
(b) 32×0.17
(c) 15.48×40

Worked Example 7

Multiplying two decimals

Without using a calculator, evaluate 12.34 × 5.6.

Solution

Method 1:

$$
\begin{aligned}
12.34 \times 5.6 &= \frac{1234}{100} \times \frac{56}{10} \\
&= \frac{1234 \times 56}{1000} \\
&= \frac{69\,104}{1000} \\
&= 69.104
\end{aligned}
$$

Method 2:

$$
\begin{array}{rrrrr}
 & 1 & 2. & 3 & 4 \\
\times & & & 5. & 6 \\
\hline
 & 7 & 4 & 0 & 4 \\
+\ 6 & 1 & 7 & 0 & \\
\hline
6 & 9. & 1 & 0 & 4 \\
\hline
\end{array}
$$

1 2 . 3 4 ← 2 decimal places (2 d.p.)

5 . 6 ← 1 decimal place (1 d.p.)

because 2 d.p. + 1 d.p

∴ 12.34 × 5.6 = 69.104

Attention

In Method 2, why do we place the decimal point at 3 d.p. in the final answer?

Hint: See Method 1.

Reflection

Which method do you prefer? Why?

Without using a calculator, find the value of each of the following.

(a) 13.56 × 2.4 (b) 137.8 × 0.351

C. Dividing decimals

In Chapter 2, we have learnt that dividing a number by a whole number or a fraction is equivalent to multiplying the number by the reciprocal of the whole number or the fraction. Let us apply this to divisions involving decimals.

Worked Example 8

Dividing decimal by whole number

Without the use of a calculator, find the value of each of the following.

(a) 2.9 ÷ 5 (b) 9.36 ÷ 24

Solution

(a) **Method 1:**

$$
\begin{aligned}
2.9 \div 5 &= \frac{29}{10} \div 5 \\
&= \frac{29}{10} \times \frac{1}{5} \\
&= \frac{29}{50} \\
&= \frac{58}{100} \\
&= 0.58
\end{aligned}
$$

Method 2:

align decimal point

$$
\begin{array}{r}
0.58 \\
5\overline{)\,2.90} \\
-0 \\
\hline
29 \\
-25 \\
\hline
40 \\
-40 \\
\hline
0
\end{array}
$$

$\therefore 2.9 \div 5 = 0.58$

(b) $9.36 \div 24$

align decimal point

$$
\begin{array}{r}
0.39 \\
24\overline{)\,9.36} \\
-0 \\
\hline
93 \\
-72 \\
\hline
216 \\
-216 \\
\hline
0
\end{array}
$$

$\therefore 9.36 \div 24 = 0.39$

Reflection

Which method do you prefer? Why?

Exercise 3B

Without the use of a calculator, evaluate the following.

(a) $0.15 \div 3$

(b) $2.7 \div 4$

(c) $3.12 \div 12$

Example 9

Dividing two decimals

Without using a calculator, find the value of $0.72 \div 0.3$.

Solution

Method 1:

$$
\begin{aligned}
0.72 \div 0.3 &= \frac{72}{100} \div \frac{3}{10} \\
&= \frac{\cancel{72}^{24}}{100} \times \frac{10}{\cancel{3}_{1}} \\
&= \frac{24}{10} \\
&= 2.4
\end{aligned}
$$

Method 2(a):

$$0.72 \div 0.3 = \frac{0.72}{0.3}$$
$$= \frac{0.72}{0.3} \times \frac{10}{10}$$
$$= \frac{7.2}{3}$$

Method 2(b):

$$0.72 \div 0.3 = \frac{0.72}{0.3}$$
$$= \frac{7.2}{3}$$

$$\begin{array}{r} 2.4 \\ 3\overline{)\,7.2} \\ -6 \\ \hline 1\;2 \\ -1\;2 \\ \hline 0 \\ \hline \end{array}$$

$\therefore 0.72 \div 0.3 = 2.4$

Attention

In Method 2(a), why can we multiply $\frac{0.72}{0.3}$ by $\frac{10}{10}$? Does it change the value of $\frac{0.72}{0.3}$?

In Method 2(b), why do we shift the decimal point in this manner?

Hint: See Method 2(a).

Without using a calculator, evaluate the following.

(a) $0.92 \div 0.4$ **(b)** $1.845 \div 0.15$

Importance of place value

Search the Internet for "25 divided by 5 = 14 video". There are three ridiculous "proofs" (of division, multiplication and addition) that show that 25 divided by 5 is 14.

Discuss what is wrong with the three "proofs".

3.3 [illegible] for length, mass and volume

In our everyday life, we encounter some units of measurement for length (e.g. centimetre, metre, kilometre), mass (e.g. gram, kilogram) and volume (e.g. millilitre, litre).

In this section, we will learn how to convert between some of these units of measurement for length, mass and volume.

A. Multiplying and dividing decimals by powers of ten

In primary school, we have learnt that the decimal point shifts when we multiply and divide numbers by 10, 100 and 1000. Why does the decimal point shift?

Copy and complete Table 3.2.

	10	100	1000
Multiplication by	$0.1 \times 10 = \frac{1}{10} \times 10$ $= 1$ $0.01 \times 10 = \square \times 10$ $= \square$ $0.001 \times 10 = \square \times 10$ $= \square$	$0.1 \times 100 = \frac{1}{10} \times 100$ $= \square$ $0.01 \times 100 = \square \times 100$ $= \square$ $0.001 \times 100 = \square \times 100$ $= \square$	$0.1 \times 1000 = \frac{1}{10} \times 1000$ $= \square$ $0.01 \times 1000 = \square \times 1000$ $= \square$ $0.001 \times 1000 = \square \times 1000$ $= \square$
	• When a decimal is **multiplied by 10**, the decimal point shifts ____ **place to the right.**	• When a decimal is **multiplied by 100**, the decimal point shifts ____ **places to the right.**	• When a decimal is **multiplied by 1000**, the decimal point shifts ____ **places to the right.**
Division by	$0.1 \div 10 = \frac{1}{10} \times \frac{1}{10}$ $= \square$ $0.01 \div 10 = \square \times \frac{1}{10}$ $= \square$ $0.001 \div 10 = \square \times \frac{1}{10}$ $= \square$	$0.1 \div 100 = \frac{1}{10} \times \frac{1}{100}$ $= \square$ $0.01 \div 100 = \square \times \frac{1}{100}$ $= \square$ $0.001 \div 100 = \square \times \frac{1}{100}$ $= \square$	$0.1 \div 1000 = \frac{1}{10} \times \frac{1}{1000}$ $= \square$ $0.01 \div 1000 = \square \times \frac{1}{1000}$ $= \square$ $0.001 \div 1000 = \square \times \frac{1}{1000}$ $= \square$
	• When a decimal is **divided by 10**, the decimal point shifts ____ **place to the left.**	• When a decimal is **divided by 100**, the decimal point shifts ____ **places to the left.**	• When a decimal is **divided by 1000**, the decimal point shifts ____ **places to the** ____.

Table 3.2

Exercise 3B

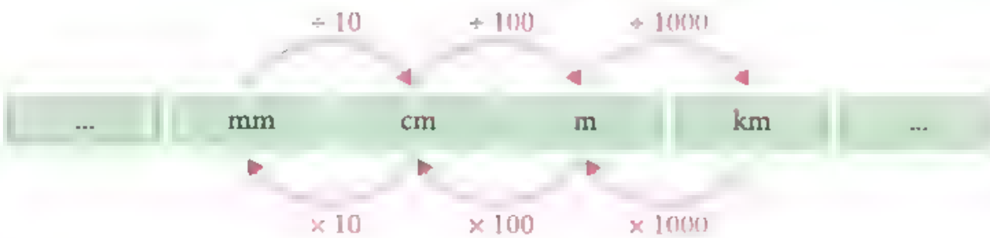
Questions

1. Without using a calculator, evaluate the following.
   (a) 0.7×10
   (b) 0.063×100
   (c) 3.61×1000
   (d) 32.9×1000

2. Without using a calculator, evaluate the following.
   (a) $0.9 \div 10$
   (b) $137 \div 100$
   (c) $143.1 \div 1000$
   (d) $2.7 \div 1000$

B. Converting units of measurement for length

Four units of measurement for length in the metric system are millimetres (mm), centimetres (cm), metres (m) and kilometres (km).



Worked Example 10

Converting unit of measurement for length

Express each of the following in the stated unit.

(a) 0.83 m in centimetres **(b)** 60 800 cm in kilometres

Solution

(a)
$$1 \text{ m} = 100 \text{ cm}$$
$$0.83 \text{ m} = 0.83 \times 100$$
$$= 83 \text{ cm}$$

(b)
$$100 \text{ cm} = 1 \text{ m}$$
$$60\,800 \text{ cm} = 60\,800 \div 100$$
$$= 608 \text{ m}$$
$$1000 \text{ m} = 1 \text{ km}$$
$$608 \text{ m} = 608 \div 1000$$
$$= 0.608 \text{ km}$$

Exercise 3B

Convert the following lengths to the stated units.

(a) 3.608 km in metres

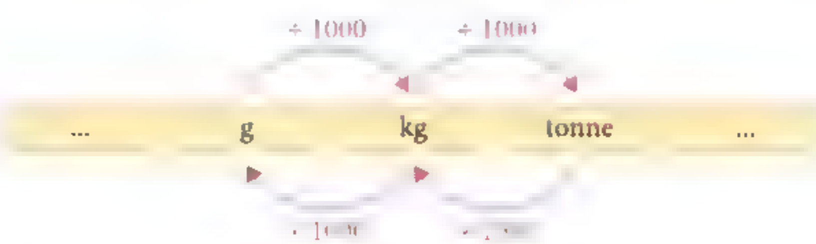
(b) 7.055 km in centimetres

(c) 1385 m in kilometres

(d) 485 mm in metres

C. Converting units of measurement for mass

We have learnt that the gram (g) and the kilogram (kg) are units of measurement for mass. Another such unit is the metric tonne (tonne).



Attention

The metric tonne (or simply, tonne) is different from the ton. A ton is an imperial unit of mass that is equivalent to 1016.047 kg, whereas the tonne is a metric unit of mass equivalent to 1000 kg.

Worked Example 11

Converting unit of measurement for mass

Express the following in the stated units.

(a) 3.25 tonnes in kilograms

(b) 9653 g in tonnes

Information

The megagram (Mg) is anothe (less common) way of expressin the same mass as the tonne, i.e. 1 megagram = 1 tonne = 1000 kilograms = 1 000 000 gram

Solution

(a) 1 tonne = 1000 kg

3.25 tonnes = 3.25 × 1000

= 3250 kg

(b) 1000 g = 1 kg

9653 g = 9653 ÷ 1000

= 9.653 kg

1000 kg = 1 tonne

9.653 kg = 9.653 ÷ 1000

= 0.009 653 tonnes

Practice Now

Express each of the following in the stated unit.

(a) 0.0575 tonnes to kilograms

(b) 3.04 tonnes in grams

(c) 6975 kg to tonnes

(d) 77 542 g to tonnes

D. Converting units of measurement for volume

A common metric unit of volume is the litre (*l*). It is related to the millilitre (ml) as shown.

Another unit of measurement for volume is the cubic metre (m^3), which will be discussed in Book 2

Worked Example 12

Converting unit of measurement for volume

Express the following in the stated units.

(a) 1200 ml in litres

(b) 3.09 litres in millilitres

Solution

(a) 1000 ml = 1 *l*

1200 ml = 1200 ÷ 1000

= 1.2 *l*

(b) $1\ l = 1000$ ml

$3.09\ l = 3.09 \times 1000$

$= 3090$ ml

Exercise 3B

Express each of the following in the stated unit.

(a) $0.95\ l$ to millilitres
(b) $5423.05\ l$ to millilitres
(c) 2765 ml to litres
(d) 89.02 ml to litres

1. How did what I learnt about multiplying and dividing fractions and mixed numbers in Chapter 2 help me in multiplying and dividing decimals?
2. How did what I learnt about multiplying and dividing decimals by powers of ten help me in converting units of measurement in the metric system?

Basic Intermediate

Exercise 3B

Do not use a calculator for this exercise.

1. Evaluate each of the following.
   (a) $2.5 + 0.2$ (b) $0.72 + 1.3$
   (c) $0.064 + 1.53$ (d) $2.12 + 0.97$
   (e) $8.28 + 3.9$ (f) $0.064 + 1.54$

2. Evaluate each of the following.
   (a) $9.5 - 8.2$ (b) $7.8 - 0.42$
   (c) $9.813 - 1.6$ (d) $9 - 8.2$
   (e) $8.03 - 4.9$ (f) $7.8 - 0.92$

3. Evaluate each of the following.
   (a) 14.72×8 (b) 0.049×9
   (c) 21×3.652 (d) 0.045×4

4. Evaluate each of the following.
   (a) 14.72×1.2 (b) 130.4×0.15
   (c) 0.27×0.08 (d) 0.25×1.963

5. Evaluate each of the following.
   (a) $0.81 \div 3$ (b) $3.426 \div 4$
   (c) $0.266 \div 8$ (d) $6.984 \div 32$

6. Find the value of each of the following.
   (a) $0.81 \div 0.3$ (b) $1.32 \div 0.12$
   (c) $3.426 \div 0.06$ (d) $4.35 \div 1.5$

7. Find the value of each of the following.
   (a) 0.63×10 (b) 40.125×100
   (c) 0.0251×1000 (d) $13.6 \div 10$
   (e) $530.5 \div 100$ (f) $26.68 \div 1000$

Intermediate | Basic

Exercise

8. Express each of the following lengths in the stated unit.
   (a) 0.123 km in metres
   (b) 83 m in centimetres
   (c) 0.1556 cm in millimetres
   (d) 1037 cm in metres
   (e) 503 m in kilometres
   (f) 2.5 mm in centimetres

9. Express each of the following masses in the stated unit.
   (a) 6.23 tonnes in kilograms
   (b) 0.066 kg in grams
   (c) 0.0256 tonnes in kilograms
   (d) 365 g in kilograms
   (e) 89 234 kg in tonnes
   (f) 2.056 kg in tonnes

10. Express each of the following in the stated unit.
    (a) 2.546 *l* in millilitres (b) 45 *l* in millilitres
    (c) 8926 ml in litres (d) 3.02 ml in litres

11. Bernard wants to deposit the coins he saved in his bank account. The number of coins in each denomination is shown in the table.

Denomination	\$0.05	\$0.10	\$0.20	\$0.50
Number of coins	300	80	60	100

How much money will Bernard deposit?

12. Express each of the following in the stated unit.
    (a) 6.15 m in metres and centimetres
    (b) 6 km 55 m in kilometres
    (c) 54.44 m in millimetres
    (d) 462.23 cm in kilometres
    (e) 89 cm 6 mm in metres
    (f) 9.123 kg in kilograms and grams
    (g) 10 kg 365 g in kilograms
    (h) 42 kg 3 g in grams

13. Ribbon A is 3.24 m longer than Ribbon B. If Ribbon A is 513.1 cm long, what is the total length of the two ribbons?

14. At a supermarket, broccoli and carrots are sold at PKR 42 per 100 g and PKR 20 per 100 g respectively. How much will 1.1 kg of broccoli and 870 g of carrots cost?

15. A glass contains 0.15 *l* of water more than a cup The total amount of water in the glass and the cup is 0.842 *l*. How much water is there in the cup? Express your answer in ml.

16. A pen and a pencil cost \$5.75 altogether. 98 pens and 8 pencils cost a total of \$156.70. How much do 20 pens and 40 pencils cost?

17. Cheryl has some coins in denominations of \$0.10, \$0.20 and \$0.50. She has 80 \$0.20 coins and 40 \$0.50 coins. If the total amount of money Cheryl has is \$143.80, how many \$0.10 coins does she have?

In this chapter, we saw that fractions and decimals are different ways of representing some numbers that lie between consecutive whole numbers We have learnt to express fractions as equivalent terminating decimals (e.g. $\frac{1}{2}$ – 0.5 and $\frac{5}{8}$ = 0.625) and vice versa. Besides terminating decimals, decimals can also be recurring, as we have seen when we express certain fractions in decimals (e.g. $\frac{1}{3}$ 0.333... and $\frac{5}{6}$ = 0.833...). Because of this, recurring decimals can also be expressed as fractions.

The equivalence of fractions and decimals means we can understand how decimals are added, subtracted, multiplied and divided using fractions.

In fact, numbers that can be expressed as a fraction of two integers, including recurring decimals, are known as rational numbers What about non-recurring, non-terminating decimals?

Summary

1. **Terminating decimals**
   (a) A *terminating decimal* is a decimal in which the digits after the decimal point terminate at some value.
   (b) A terminating decimal can be expressed as a fraction with a power of ten as the denominator, and vice versa
   Example:
   $0.5 = \frac{5}{10} = \frac{1}{2}$
2. **Recurring decimals**
   (a) A *recurring decimal* contains digits that repeat indefinitely. Recurring decimals can be written with a dot (or a bar) above the repeating digit(s).
   Examples:
   $0.333\ldots = 0.\dot{3}$
   $0.451\ 451\ 451\ldots = 0.\dot{4}5\dot{1}$
   $0.123\ 232\ 322\ldots = 0.1\dot{2}\dot{3}$
   - Give two examples of a recurring decimal and express each of them using dot notation.

   (b) Long division is used to convert a fraction to a recurring decimal.
   Example:

$$\begin{array}{r|l} & \;\;0.4\,5\,4 \\ 11 & \;\;5.0\,0\,0 \\ & -4\,4 \\ \hline & \;\;\;\;6\,0 \\ & \;\;\;\;5\,5 \\ \hline & \;\;\;\;\;\;5\,0 \\ & -\;\;\;4\,4 \\ \hline & \;\;\;\;\;\;\;\;6 \end{array}$$

   $\therefore \frac{5}{11} = 0.\dot{4}\dot{5}$
   - Write down two other fractions that can be expressed as recurring decimals.

(c) When converting a recurring decimal to a fraction,

(i) multiply the decimal by a power of 10, repeating with another power of 10 where necessary;

(ii) subtract the results obtained in the step above to eliminate the recurring digits; and

(iii) divide the result from the second step by the difference between the powers of 10.

Example:

(i) Let x be $0.\dot{3}\dot{2}$.

$$100x = 32.\dot{3}\dot{2}$$

(ii) $100x - x = 32.\dot{3}\dot{2} - 0.\dot{3}\dot{2}$

$$99x = 32$$

(iii) $x = \frac{32}{99}$

3. **Addition and subtraction of decimals**

When adding or subtracting decimals, we align the decimal points.

Example:

$$\begin{array}{r} \overset{1}{2}.60 \leftarrow \text{insert a zero in the hundredths place as a place holder} \\ +\ 6.78 \\ \hline 9.38 \\ \hline \end{array}$$

(align decimal point)

4. **Multiplication of decimals**

(a) When multiplying decimals by an integer, the final answer will have the same number of decimal places as the original decimal.

Example:

$$\begin{array}{r} 3.\overset{1}{3}6 \leftarrow \text{2 decimal places (2 d.p.)} \\ \times\ \ 12 \\ \hline 672 \\ 3360 \\ \hline 40.32 \leftarrow \text{final answer has 2 d.p.} \\ \hline \end{array}$$

$\therefore 3.36 \times 12 = 40.32$

(b) When multiplying two decimals, the number of decimal places in the final answer will be the sum of the number of decimal places of the two original decimals.

Example:

$$\begin{array}{r} 12.34 \leftarrow \text{2 decimal places (2 d.p.)} \\ \times\ \ 5.6 \leftarrow \text{1 decimal place (1 d.p.)} \\ \hline 7404 \\ +\ 6170\ \ \\ \hline 69.104 \\ \hline \end{array}$$

$\therefore 12.34 \times 5.6 = 69.104$

5. **Division of decimals**

(a) When dividing decimals by an integer, the decimal point in the quotient must align with the decimal point of the dividend.

Example:

align decimal point

$$\begin{array}{r} 0.58 \\ 5\overline{)\,2.90} \\ -0 \\ \hline 29 \\ -25 \\ \hline 40 \\ -\;40 \\ \hline 0 \\ \hline \end{array}$$

$\therefore\ 2.9 \div 5 = 0.58$

(b) When dividing two decimals, first multiply both dividend and divisor by the same power of ten such that the divisor becomes a whole number. Then perform the same step as outlined in point 5(a) above.

Example:

$$\begin{aligned} 0.72 \div 0.3 &= \frac{0.72}{0.3} \\ &= \frac{0.72}{0.3} \times \frac{10}{10} \\ &= \frac{7.2}{3} \end{aligned}$$

align the decimal points

$$\begin{array}{r} 2.4 \\ 3\overline{)\,7.2} \\ -6 \\ \hline 1\;\;2 \\ -1\;\;2 \\ \hline 0 \\ \hline \end{array}$$

$\therefore\ 0.72 \div 0.3 = 2.4$

6. **Multiplying and dividing decimals by 10, 100 or 1000**

(a) When multiplying or dividing decimals by 10, 100 or 1000, the decimal point shifts to the right or left respectively.

Examples:

$0.01 \times 100 = 1$ $\qquad\qquad$ $1.0 \div 1000 = 0.001$

(b) The multiplication and division of decimals by 10, 100 or 1000 are applied when converting units in the metric system, such as the units of mass (g, kg, tonne), length (mm, cm, m, km) and volume (ml and *l*).

Examples:

$$\begin{aligned} 0.83\text{ m} &= 0.83 \times 100 \\ &= 83\text{ cm} \\ 1200\text{ ml} &= 1200 \div 1000 \\ &= 1.2\ l \end{aligned}$$

CHAPTER 4

Integers, Rational Numbers and Real Numbers

The coldest continent on Earth is Antarctica, where temperatures range from 10 °C in summer to −80 °C (read *negative* 80 degrees Celsius) in winter. '−80 °C' represents 80 °C below 0 °C.

Altitude refers to the height of an object using sea level as a reference. Mount Everest, the highest mountain on Earth, has an altitude of 8848 m above sea level. The Dead Sea has an altitude of −430 m (read *negative* 430 metres). This signifies that the shores of the Dead Sea are 430 m below sea level.

Are there other real-life examples of negative numbers that you can think of?

In this chapter, we will learn how to perform operations (e.g. add or subtract) on negative numbers, and discover other types of numbers called rational numbers and real numbers.

Learning Outcomes

What will we learn in this chapter?

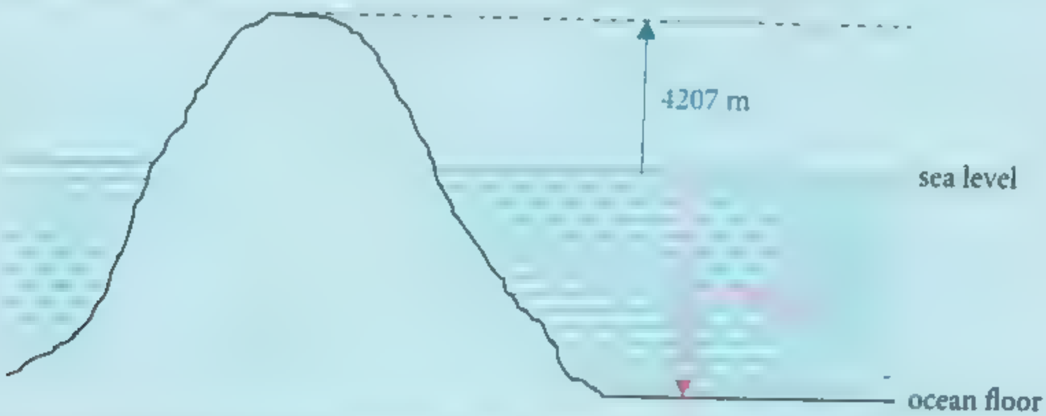
- What negative, rational and real numbers are
- How to order these numbers and perform operations on them
- Why these numbers have useful applications in real-world contexts

Introductory Problem

In the Chapter Opener, we learnt that Mount Everest is the highest mountain on Earth and its altitude (or height) is 8848 m above sea level. However, it is not the tallest mountain.

Mauna Kea, a volcanic mountain on the island of Hawaii, has an altitude of 4207 m above sea level, and appears shorter than Mount Everest.

However, the height of a mountain is measured from its base. Part of Mauna Kea is submerged underwater, and hence its total height from the ocean floor is more than what appears above sea level.



As the ocean floor is not level, we can only estimate that the altitude of the ocean floor is over −5793 m.

(i) Find the minimum height of Mauna Kea.

(ii) Which is taller? Mount Everest or Mauna Kea?

In this chapter, we will learn how to solve such problems involving operations on negative numbers, rational numbers and real numbers.

4.1 Negative numbers

We have learnt about whole numbers, decimals and fractions, e.g. 0, 7, 1.6 and $\frac{1}{2}$. These numbers are greater than or equal to 0. Numbers that are greater than 0 are called **positive numbers**.

Negative numbers are numbers that are less than 0, such as −80, −1.4, $-\frac{3}{5}$.

> **Attention**
>
> 'Negative' is different from 'minus'. 'negative' refers to the *state* of the number, while 'minus' refers to the *operation* of subtraction.

As we have seen in the Chapter Opener and the Introductory Problem, we do encounter negative numbers in the real world, e.g. −80 °C and −5793 m. The negative number, −80, is read as *negative* 80.

When measuring temperature, 0 °C is the **reference** temperature. Temperatures above 0 °C are positive while temperatures below 0 °C are negative. A temperature of −80 °C means that the temperature is 80 °C below 0 °C

What is the reference level when measuring altitudes?

Uses of negative numbers in real-world contexts

Brainstorm a few more examples of negative numbers in the real world.
For each example, explain the meaning of the negative number, e.g. –80 °C means 80 °C below 0 °C.

A number is made up of an **absolute value** with a positive or negative sign in front
The negative number –80 has an absolute value of 80 and a negative sign in front.

We usually write a positive number such as '+80' as '80' without the positive sign.
What is the absolute value of the positive number 80?

–80
absolute valu

A. Integers

In primary school, we have learnt about whole numbers: 0, 1, 2, 3, …
What about numbers such as –1, –2, –3, –4, …?
Together, these numbers are called **integers**.

…, –3, –2, –1, 0, 1, 2, 3, …
negative integers   positive integers

Practise Now 1A

1. From the given numbers,

$-5,\ 2020,\ 0,\ -\frac{1}{2},\ 1.666,\ -3.8,\ \frac{3}{4},\ -17,\ 6,\ -\frac{5}{3}$

list the numbers that are

(i) positive integers, **(ii)** negative integers,
(iii) positive numbers, **(iv)** negative numbers.

2. **(a)** The coldest temperature ever recorded in Korea was 43.6 °C below 0 °C in the winter of 1933 at Junggangjin. Represent this temperature using a negative number.

(b) The lowest known point on Earth is the Challenger Deep in the Mariana Trench in the western Pacific Ocean. It is 10 994 m below sea level. Represent this altitude using a negative number.

(c) A company suffers a loss or a deficit of $10 000 in the year 2023. Represent this loss using a negative number.

(d) If 45° represents a clockwise rotation of 45°, represent an anticlockwise rotation of 81° using a negative number.

B. Number line

Ordering of Numbers

Fig. 4.1 shows a thermometer.

1. **(a)** What are the temperatures indicated by each of the points *A*, *B* and *C*?
(b) Which point shows the highest temperature?
(c) Which point shows the lowest temperature?

Fig. 4.1

The markings on a thermometer enable us to read and compare temperatures. Similarly, we can represent numbers on a diagram called a **number line** (see Fig. 4.2). The markings are equally spaced and the number 0 is the *reference*. The arrow pointing right indicates the direction in which numbers increase.

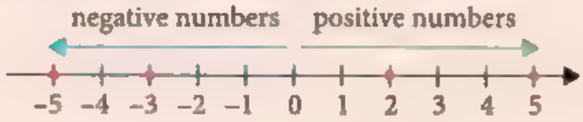


Fig. 4.2

Diagrams
The number line is used to represent numbers and allows us to visualise how they are related to each other.

Information
In some countries, the number line is drawn with an arrow at each end to indicate that the line goes on indefinitely in both directions.
You can also draw a number line vertically.

2. Copy and complete the following:

(a) The numbers on the right of 0 are positive numbers.

(b) The numbers on the left of 0 are ______ numbers.

(c) A number that is on the right of another number is *more than (or greater than)* that number.
For example, 5 is on the right of 2, so 5 is ______ than (or ______ than) 2.

(d) A number which is on the left of another number is *less than (or smaller than)* that number.
For example, 2 is on the left of 5, so 2 is ______ than (or ______ than) 5.

We have encountered the symbol >, which represents *more than (or greater than)*, and the symbol <, which represents *'less than' (or 'smaller than')*.

The symbol ⩾ represents *more than or equal to*, and the symbol ⩽ represents *'less than or equal to'*.

Notations
Notations, such as >, <, ⩾ and ⩽, help to represent the relationships between numbers concisely and precisely.

3. Look at the number line in Fig. 4.2 and answer each of the following questions. Use '>' or '<' to represent the relationship between the two numbers in each question and explain your answer.

(a) Is −3 more or less than 2?

(b) Is −3 more or less than −5?

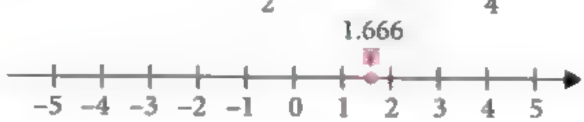
1. Fill in each box with '>' or '<'. The first one has been done for you.

(a) −3 > −5 **(b)** −7 ☐ −2

(c) 4 ☐ −4 **(d)** −6 ☐ −100

2. Use a dot to represent each of the following numbers on the number line below. An example is given.

$$-5,\ 4,\ 0,\ -1\frac{1}{2},\ 1.666,\ -3.8,\ \frac{3}{4}$$



Hence, arrange the numbers in ascending order, i.e. from the smallest to the greatest.

1. What do I already know about negative numbers in the real world that could help me to order positive and negative numbers on a number line?
2. What have I learnt in this section that I am still unclear of?

Intermediate | Basic

Exercise

1. $-0.3, \frac{1}{5}, 0, -\frac{9}{7}, 4.33, -12, 10\,001, -1\frac{1}{2}, -2026, 4$
   From the given numbers, list the numbers that are
   (i) positive integers, (ii) negative integers,
   (iii) positive numbers, (iv) negative numbers.

2. (a) Absolute zero, defined as 0 Kelvin, is the theoretical lowest possible temperature. 0 Kelvin is 273.15 °C below 0 °C. Represent this temperature using a negative number.
   (b) The lowest point in North America is the Badwater Basin which is 86 m below sea level. Represent this altitude using a negative number.
   (c) An investment portfolio suffers a loss of \$85 000 in the year 2020. Represent this loss using a negative number.
   (d) A baby has lost a mass of 0.15 kg. Represent this loss in mass using a negative number

3. Fill in each box with '>' or '<'.
   (a) 16 ☐ 60 (b) 3.1 ☐ 3.2
   (c) −6 ☐ 8 (d) 30 ☐ −31
   (e) −2 ☐ 0 (f) 9.8 ☐ −9.9

4. Use a number line to illustrate each of the following.
   (a) $2\frac{2}{5}, 0, -4, 6, -2.8$
   (b) $-0.55, 4, -\frac{1}{10}, 2, -2$
   (c) integers between −4 to 4
   (d) positive integers less than 10

5. Using a number line, arrange each of the following in ascending order.
   (a) 230, −13, 23, −3, 30
   (b) $-0.5, 150, 15, -10, -\frac{3}{20}$

6. (a) If 6 m represents 6 m above sea level, what does −10 m represent?
   (b) If +\$40 represents depositing \$40 in the bank, what does −\$25 represent?
   (c) If −60° represents a clockwise rotation of 60°, what does +30° represent?
   (d) If a score of +2 represents an increment of 2 marks in a test, what does a score of −4 represent?

7. Fill in each box with '>' or '<'.
   (a) −4 ☐ −6 (b) −11 ☐ −11.5
   (c) $\frac{1}{5}$ ☐ $\frac{1}{3}$ (d) $-\frac{1}{3}$ ☐ $-\frac{5}{6}$

8. Use a number line to illustrate each of the following.
   (a) $-\frac{1}{3}, 2.5, 1\frac{3}{8}, 1, -0.2, 0.11$
   (b) Positive odd integers less than 20
   (c) Prime numbers more than or equal to 2 but less than 10
   (d) Common positive factors of 12 and 16

Basic | Intermediate | Advanced

Exercise 4A

9. (a) Write down two integers less than -100.
(b) Write down three non-negative integers greater than -10.
(c) Write down four integers between -7 and 0 inclusive.
Hint 'Between -7 and 0 *inclusive*' means the numbers *can* be -7 or 0.
(d) Write down two numbers between -5 and -4.
Hint 'Between -5 and -4' means the numbers *cannot* be -5 or -4.

10. Bernard said, "Since 7 is greater than 4, then -7 should also be greater than -4."
Is this statement true or false? Explain.

11. The boiling point of alcohol is 80 °C and the boiling point of water is 100 °C. When a mixture of alcohol and water is heated to a temperature of 95 °C, only water will remain.
The boiling points of liquid nitrogen, liquid xenon and liquid oxygen are -196 °C, -108 °C and -183 °C respectively. A mixture of liquid nitrogen, xenon and oxygen was kept at a temperature of -215 °C. The mixture is then warmed to a temperature of -185 °C. Which of the liquids will turn into a gas? Explain.

4.2 Addition and subtraction involving negative integers

In primary school, we have learnt how to subtract a smaller positive number from a greater positive number, e.g. $5 - 2 = 3$.
How do we subtract a greater positive number from a smaller positive number, e.g. $2 - 5$?
In this section, we will learn how to add and subtract numbers that involve negative integers using number discs.
Fig. 4.3 shows number discs representing 1 (i.e. +1) and -1 respectively:

Fig. 4.3

To represent the number 2, we use two discs as shown:

To represent the number -2, we use two discs as shown.

Just For Fun

Search the Internet for KenKen, a puzzle created by Japanese mathematics teacher Tetsuya Miyamoto. It is somewhat like Sudoku, but it involves the four operations on positive numbers. You can also download any app on KenKen to play. Have fun!

A. Addition of two negative integers

What is (−2) + (−3) equal to?

To represent the number −2, we put two (−1) discs as shown in Fig. 4.4(a).

To add the number −3, we add three more (−1) discs as shown in Fig. 4.4(b).

Add 3 more
(−1) discs

(a) (b)

Fig. 4.4

∴ (−2) + (−3) = −5

> **Attention**
> (−2) + (−3) can be written as −2 + (−3). The first pair of brackets is not necessary, but th[e] second pair is necessary. Why?

Addition of two negative integers

Work out the answers using number discs as shown previously. Write down the final answer for each part.

(a) **(i)** (−4) + (−3)   **(b)** **(i)** (−5) + (−1)
  (ii) 4 + 3   **(ii)** 5 + 1

For parts **(a)** and **(b)**, what do you observe about
- the absolute value of the result in **(i)** and **(ii)**?
- the sign of the result in **(i)** and **(ii)**?

From the above Investigation, we observe the following:

> **Addition of two negative integers**
> Adding two negative integers gives the same result as the [illegible] [illegible]. E.g. (−2) + (−3) = −(2 + 3) = −5

Addition of two negative integers

Without using a calculator, evaluate (−6) + (−2).

Solution

(−6) + (−2) = −8

> **Problem-solving Tip**
> (−6) + (−2) = −(6 + 2)
> = −8
> However, there is no need to write the intermediate step −(6 + 2) in your working.

Without using a calculator, evaluate the following.

(a) (−4) + (−5)   **(b)** −9 + (−7)

(c) (−21) + (−73)   **(d)** (−67) + (−48)

B. Zero pair

What is the overall value if we put one (1) disc and one (−1) disc together as shown in Fig. 4.5?

Fig. 4.5

> **Attention**
> Suppose you have \$1 and you owe a friend \$1 (i.e. −\$1).
> In actual fact, your balance is \$0.

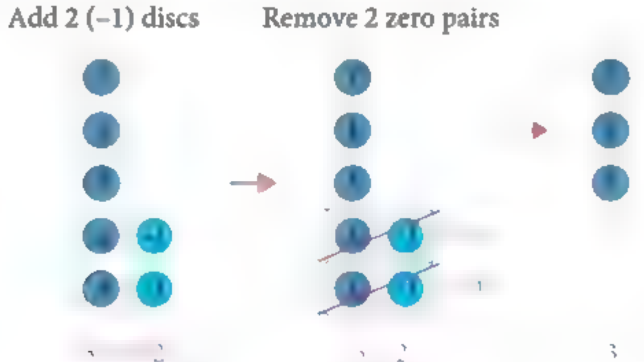
The overall value is 0 and we call this a **zero pair**.

C. Addition of a positive and a negative integer

What is 5 + (−2) equal to?

To represent 5 + (−2), we put five (1) discs and two (−1) discs as shown in Fig. 4.6(**a**).

From Fig. 4.6(**b**), we see that 2 zero pairs are formed. Since zero pairs are equal to 0, they are removed as shown in Fig. 4.6(**c**).



(a) **(b)** **(c)**

Fig. 4.6

> **Attention**
> Removing the two zero pairs leads to removing two (1) discs from the five (1) discs.
> In other words, 5 + (−2) = 5 − 2.

$\therefore\ 5 + (-2) = 5 - 2$

$\qquad\qquad\ = 3$

Addition of a positive and a negative integer

Work out the answers using number discs as shown above. Write down the final answer for each part.

(a) **(i)** 6 + (−2) **(b)** **(i)** −3 + 4

(ii) 6 − 2 **(ii)** 4 − 3

For parts **(a)** and **(b)**, what do you observe about the results in **(i)** and **(ii)**?

> **Attention**
> −3 + 4 is the same as (−3) + 4 which is equal to 4 + (−3).

From the above Investigation, we observe the following:

> **Addition of a positive and a negative integer**
> Adding a negative integer gives the same result as [illegible]. E.g. 5 + (−2) = 5 − 2 = 3 and −2 + 5 = 5 − 2 = 3

Example 2

Addition of a positive and a negative integer

Without using a calculator, evaluate the following.

(a) $7 + (-6)$ (b) $-8 + 9$

Solution

(a) $7 + (-6) = 7 - 6$
$= 1$

(b) $-8 + 9 = 9 - 8$
$= 1$

Exercise 4B

Without using a calculator, evaluate the following.

(a) $8 + (-3)$ (b) $45 + (-17)$

(c) $-2 + 6$ (d) $-12 + 35$

D. Subtraction between two positive integers

We can use number discs to find out the value of $2 - 5$.

To represent the number 2, we put two discs as shown in Fig. 4.7(**a**). There are not enough discs to subtract or *take away* 5, so we add three zero pairs as shown in Fig. 4.7(**b**). But why can we just add three zero pairs? Because $2 + 0 + 0 + 0$ is still equal to 2.

There are now enough discs to subtract or take away as shown in Fig. 4.7(**c**) and (**d**).

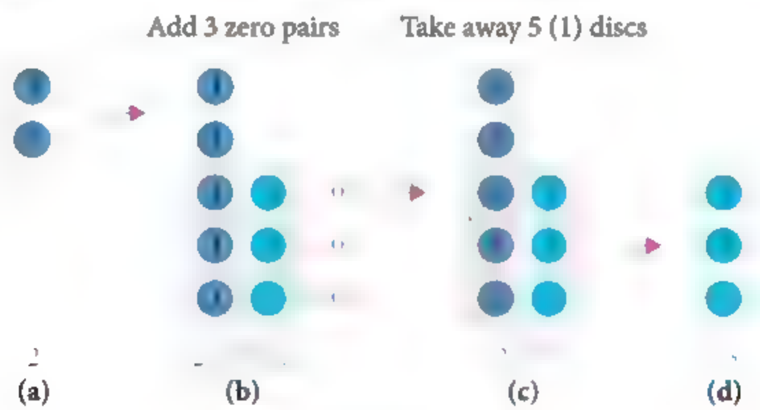


Fig. 4.7

$\therefore 2 - 5 = -3$

The answer is negative because we are subtracting a larger value from a smaller value.

Work out the answers using number discs as shown above. Write down the final answer for each part.

(a) (i) $4 - 3$ **(b)** (i) $6 - 1$

(ii) $3 - 4$ (ii) $1 - 6$

For parts **(a)** and **(b)**, what do you observe about

- the absolute value of the result in (i) and (ii)?
- the sign of the result in (i) and (ii)?

From the above Investigation, we observe the following:

Subtraction between two positive integers

Subtracting a greater positive integer from a smaller positive integer gives the same result as the negative of the difference of the two integers. E.g. $2 - 5 = -(5 - 2) = -3$

Subtraction between two positive integers

Without using a calculator, find the value of each of the following.

(a) $4 - 7$ **(b)** $12 + (-23)$

Solution

(a) $4 - 7 = -3$

(b) $12 + (-23) = 12 - 23$ recall: $5 + (-2) = 5 - 2$

$= -11$

Problem-solving Tip

(a) $4 - 7 = -(7 - 4)$

$= -3$

However, there is no need to write the intermediate step $-(7 - 4)$ in your working.

Without using a calculator, calculate the value of each of the following.

(a) $5 - 9$ **(b)** $38 - 59$

(c) $8 + (-11)$ **(d)** $(-92) + 47$

E. Subtraction of a positive integer from a negative integer

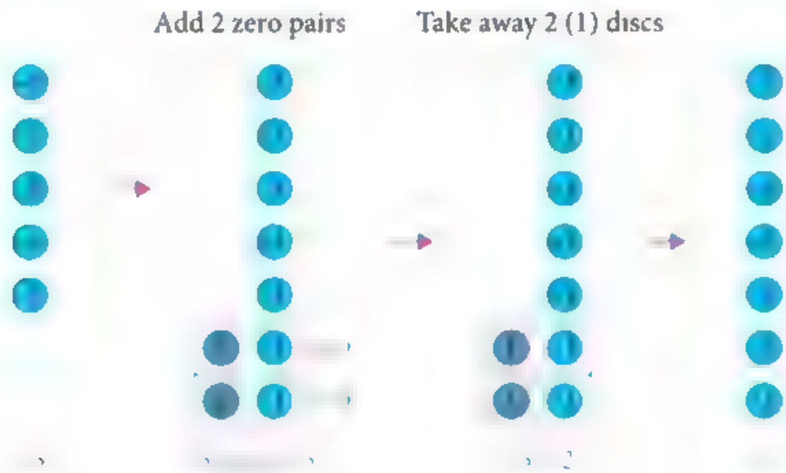
What is $-5 - 2$ equal to?

To represent the number -5, we put five discs as shown in Fig. 4.8**(a)**.

Since there are no positive discs to subtract or take away, we add two zero pairs as shown in Fig. 4.8**(b)**. But why can we just add two zero pairs?

Because $-5 + 0 + 0$ is still equal to -5.

There are now enough discs to subtract or take away as shown in Fig. 4.8**(c)** and **(d)**.



(a) (b) (c) (d)

Fig. 4.8

$\therefore -5 - 2 = -7$

Manipulating the number discs in Fig. 4.8(c), we see that $-5 - 2 = -5 + (-2)$. This is consistent with what we have observed in Section 4.2C, i.e. $5 + (-2) = 5 - 2$.

Recall also from Section 4.2A that $-5 + (-2) = -(5 + 2) = -7$, just like $-5 - 2 = -7$, so $-5 - 2 = -(5 + 2) = -7$.

Subtraction of a positive integer from a negative integer

Work out the answers to parts **(a)** and **(b)** below, using number discs as shown previously. Write down the final answer for each part.

(a) $-4 - 1$ **(b)** $-3 - 6$

For each part, what do you observe about the result when compared to the sum of the absolute values of the integers?

From the above Investigation, we observe the following:

> **Subtraction of a positive integer from a negative integer**
> Subtracting a positive integer from a negative integer gives the same result as the [illegible] of the two integers. E.g. $-5 - 2 = -(5 + 2) = -7$

Worked Example 4

Subtraction of a positive integer from a negative integer

Without using a calculator, find the value of $-8 - 3$.

Solution

$-8 - 3 = -11$

Problem-solving Tip

$-8 - 3 = -(8 + 3)$
$= -11$

Practise Now 4

Similar and Further Questions
Exercise 4B
Questions 4(a)–(d)

Without using a calculator, find the value of each of the following.

(a) $-9 - 11$ **(b)** $-39 - 12$ **(c)** $-146 - 218$

F. Subtraction of a negative integer

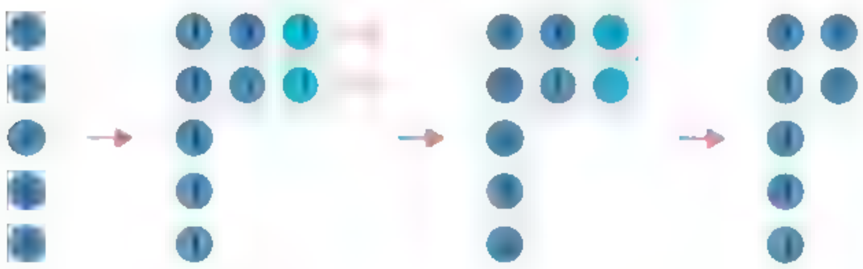
What is $5 - (-2)$ equal to?

To represent the number 5, we put five discs as shown in Fig. 4.9(**a**), but there are no discs to subtract or take away -2.

So we add two zero pairs as shown in Fig. 4.9(**b**). We add two zero pairs because $5 + 0 + 0$ is still equal to 5.

Now we have two discs to subtract or take away as shown in Fig. 4.9(**c**) and (**d**).

Add 2 zero pairs   Take away 2 (−1) discs



5	5 + 0 = 5	5 − (−2)	5 + 2 = 7
(a)	**(b)**	**(c)**	**(d)**

Fig. 4.9

Attention

Manipulating the number discs from Fig. 4.9(**b**) to 4.9(**c**), we see that $5 - (-2) = 5 + 2$.

$\therefore 5 - (-2) = 5 + 2$
$\quad\quad\quad\quad = 7$

Subtraction of a negative integer

Work out the answers using number discs as shown previously. Write down the final answer for each part.

(a) **(i)** $6 - (-2)$ **(b)** **(i)** $4 - (-3)$
  (ii) $6 + 2$ **(ii)** $4 + 3$

For parts **(a)** and **(b)**, what do you observe about the results in **(i)** and **(ii)**?

From the above Investigation, we observe the following:

Subtraction of a negative integer

Subtracting a negative integer gives the same result as [illegible]. E.g. $5 - (-2) = 5 + 2 = 7$

Worked Example 5

Subtraction of a negative integer

Without using a calculator, evaluate the following.

(a) $2 - (-7)$ (b) $-5 - (-4)$

Solution

(a) $2 - (-7) = 2 + 7$
$= 9$

(b) $-5 - (-4) = -5 + 4$
$= 4 - 5$ recall: $-2 + 5 = 5 - 2$
$= -1$ recall: 2

Information

(a) involves subtracting a negative number from a positive number while (b) involves two negative numbers. In Worked Example 4, we learnt to subtract a positive number from a negative number.

Practise Now 5

Exercise 4B

Without using a calculator, find the value of each of the following.

(a) $4 - (-6)$ (b) $30 - (-16)$
(c) $-8 - (-3)$ (d) $-17 - (-35)$

Worked Example 6

Subtraction of a negative integer in real-world context

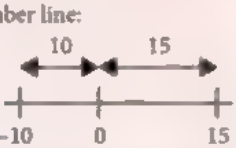
The figure shows a thermometer. The readings are in °C. Find the difference between the temperatures indicated by the points *A* and *B*.

Solution

Point *A* shows −10 °C.
Point *B* shows 15 °C.
Difference in temperature $= 15\text{ °C} - (-10\text{ °C})$
$= 15\text{ °C} + 10\text{ °C}$
$= 25\text{ °C}$

Attention

Subtracting a negative integer, e.g. $15 - (-10)$, can be interpreted as the *difference* between −10 and 15 on a number line:

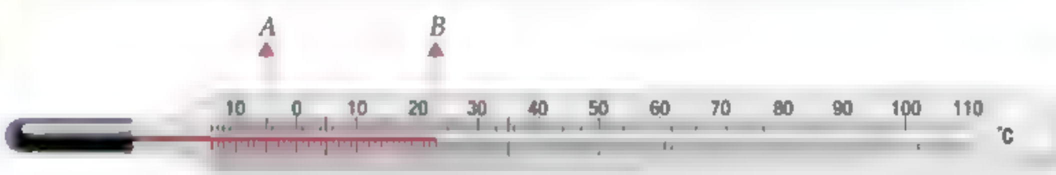


Can you see that the difference is $10 + 15$?

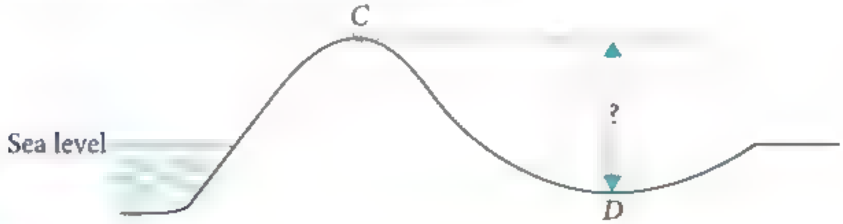
Practise Now 6

Exercise 4B
Questions _, 7, 14, 16

1. The figure shows a thermometer. The readings are in °C.
Find the difference between the temperatures indicated by the points *A* and *B*.



2. A holiday resort C is located at the top of a hill which is 314 m above sea level. A tourist attraction D lies at the bottom of a valley which is 165 m below sea level. Represent the altitude at D using a negative number. Hence, find the difference in altitude between the holiday resort and the tourist attraction.



Introductory Problem Revisited

In the Introductory Problem, you may not have known how to subtract a negative number to find the minimum height of Mauna Kea, i.e. 4207 − (−5793) m. Are you able to do so now? Discuss this with your classmates.

G. Summary of addition and subtraction of integers

Addition of two negative integers:	e.g. $(-2) + (-3) = -(2 + 3) = -5$
Addition of a positive and a negative integer:	e.g. $5 + (-2) = 5 - 2 = 3$ and $-2 + 5 = 5 - 2 = 3$
Subtraction between two positive integers:	e.g. $2 - 5 = -(5 - 2) = -3$
Subtraction of a positive integer from a negative integer:	e.g. $-5 - 2 = -(5 + 2) = -7$
Subtraction of a negative integer:	e.g. $5 - (-2) = 5 + 2 = 7$

Puzzle for consolidation

Why should we not have a conversation near the Merlion in Singapore? Find the value of each of the following and write the letter in the box above/below the answer to find out.

A $-5 - 6$	**T** $0 + (-8)$	**P** $-47 + 16$	**O** $5 - 27$
S $0 - (-4)$	**Y** $-88 + 70$	**N** $-38 - 10$	**E** $2 - 9$
D $5 + (-11)$	**R** $-6 - (-17)$	**U** $9 - (-14)$	**H** $8 + (-6)$
W $-7 - 9$			

−7   −31   −11   4   −11   −7   −22   −18   23   2   −11

2   4   11   −18   −16   −8   11   −48   −22   11   −7   −6

Similar and Further Questions
Exercise 4B
Questions 5(a)–(d), 7(a), (b)

1. What do I already know about the addition and subtraction of positive integers that could guide my learning of the addition and subtraction of negative integers?
2. Where do I make mistakes when dealing with addition or subtraction involving negative numbers? What is my confusion and how can I overcome it?

Intermediate | Basic

Exercise

1. Evaluate the following.

(a) $(-8) + (-3)$ (b) $-1 + (-6)$
(c) $(-4) + (-7)$ (d) $-5 + (-2)$

2. Evaluate the following.

(a) $9 + (-4)$ (b) $10 + (-3)$
(c) $-17 + 29$ (d) $-20 + 20$

3. Calculate the value of each of the following.

(a) $3 - 8$ (b) $0 - 11$
(c) $7 - 14$ (d) $5 + (-8)$
(e) $0 + (-19)$ (f) $-12 + 7$

4. Find the value of each of the following.

(a) $-2 - 7$ (b) $-8 - 5$
(c) $-4 - 9$ (d) $-6 - 3$

5. Evaluate the following.

(a) $3 - (-9)$ (b) $6 - (-5)$
(c) $-8 - (-2)$ (d) $-4 - (-7)$

6. The figure shows part of a thermometer. The readings are in °C. Find the difference between the temperatures indicated by the points *A* and *B*.

7. The temperature of a city on a particular night is −5 °C. The next morning, the temperature rises by 3 °C. Find the temperature in the morning.

8. What is Ken's favourite day? Find the value of each of the following and write the letter in the box above/below the answer to find out.

A	$7 + (-3)$	**B**	$19 - (-1)$	**D**	$-5 - (-5)$
E	$8 - (-8)$	**H**	$-14 + 6$	**I**	$-2 - (-9)$
L	$-1 + (-6)$	**M**	$3 + (-12)$	**N**	$8 - (-2)$
O	$-4 + 6$	**T**	$-10 - (-5)$	**Y**	$-3 - 3$

2 7 4

8 7 0 6

9. Find the value of each of the following.

(a) $(-14) + (-16)$ (b) $-52 + (-27)$
(c) $(-39) + (-65)$ (d) $-138 + (-22)$

10. Calculate the value of each of the following.

(a) $47 + (-13)$ (b) $137 + (-24)$
(c) $-95 + 113$ (d) $-78 + 139$

11. Evaluate the following.

(a) $16 - 69$ (b) $14 - 76$
(c) $88 + (-123)$ (d) $76 + (-183)$
(e) $-73 + 26$ (f) $-111 + 12$

Exercise

12. Evaluate the following.

(a) $-84 - 23$ (b) $-69 - 76$

(c) $-714 - 716$ (d) $-767 - 697$

13. Find the value of each of the following.

(a) $24 - (-11)$ (b) $38 - (-57)$

(c) $-69 - (-28)$ (d) $-34 - (-91)$

14. A city is located at a height of 138 m above sea level while a town is at a height of 51 m below sea level. Represent the altitude of the town using a negative number. Hence, find the difference in altitude between the city and the town.

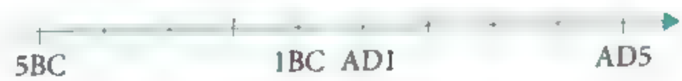
15. Without calculating the value of each of the following, explain whether each value is positive or negative.

(a) $(-987) + (-654)$ (b) $123 - 456$

(c) $436 + (-634)$ (d) $185 - (-567)$

(i) Using the number line, find the difference between -2 and 3.

(ii) The figure below shows a timeline for BC and AD. For example, 2 BC stands for 2 years Before Christ and AD 3 stands for 3 years Anno Domini (which means In the year of the Lord, i.e. after Christ was born).



What is the main difference between the timeline for BC and AD, and the number line?

(iii) How many years are there between 2 BC and AD 3?

(iv) Think of another real-life example that is similar to the timeline in part **(ii)** but different from the number line in part **(i)**.

OPEN

(a) Give two examples of a pair of numbers, x and y, such that $x + y = -10$.

(b) Give two examples of a pair of numbers, x and y, such that $x - y = -10$.

4.3 Multiplication, division and combined operations involving negative integers

In primary school, we have learnt how to multiply positive numbers, e.g. 2×3.

How do we multiply a positive number by a negative number, or two negative numbers together, e.g. $2 \times (-3)$, $(-3) \times 2$ or $(-3) \times (-2)$?

In this section, we will learn how to multiply numbers that involve negative integers using number discs.

A. Negative of an integer

In Section 4.2, we have learnt how to represent 1 (i.e. +1) and −1 using and respectively.

If we flip the disc, you will realise that the back of the disc is actually .

In other words, the *negative of 1* can be obtained by flipping the disc to obtain , and we represent this as follows:

Fig. 4.10

How do we obtain the *negative of (−1)*, i.e. what is −(−1)?

By flipping the disc as shown below, we obtain −(−1) = 1.

Fig. 4.11

Negative of an integer

Work out the answers to parts **(a)** and **(b)** below, using number discs as shown above. Write down the final answer for each part.

(a) Negative of 3 **(b)** Negative of (−3), i.e. −(−3)

B. Multiplication involving negative integers

Case 1: 2 × 3

The product 2 × 3 means 2 groups of 3, and it can be represented by number discs as shown in Fig. 4.12, i.e. 2 × 3 = 6.

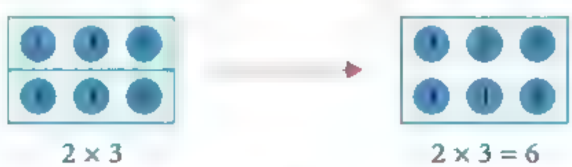


Fig. 4.12

Case 2: 2 × (−3)

The product 2 × (−3) means 2 groups of −3, and it can be represented as shown in Fig. 4.13, i.e. 2 × (−3) = −6.

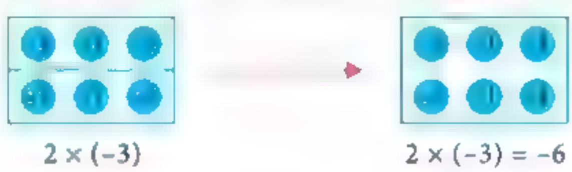


Fig. 4.13

Case 3: (−3) × 2

What does the product (−3) × 2 mean?

Does it mean 'negative 3 groups' of 2? But what does 'negative 3 groups' mean?

Since 2 × 3 = 3 × 2, i.e. we can change the order of the numbers in a product, then (−3) × 2 = 2 × (−3) = −6 from Case 2.

How do we get −6 from (−3) × 2 directly?

Since −6 = −(3 × 2), we can *interpret (−3) × 2 as −(3 × 2)*, i.e. (−3) × 2 means the negative of '3 groups of 2'.

So we flip 3 groups of 2 as shown in Fig. 4.14 to get (−3) × 2 = −6.

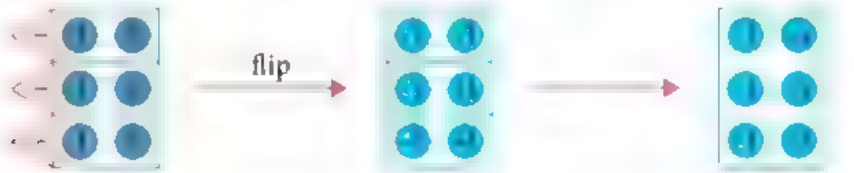


Fig. 4.14

> **Attention**
> (−3) × 2 can be written as −3 × 2, i.e. the brackets are not necessary. In other words, we can also write: −3 × 2 = −6. The middle diagram in Fig. 4.14 shows that (−3) × 2 = 3 × (−2) as well.

Case 4: (−3) × (−2)

What does the product (−3) × (−2) mean?

From Case 3, we have interpreted (−3) × 2 as −(3 × 2).

Similarly, we can *interpret (−3) × (−2) as −[3 × (−2)]*, i.e. (−3) × (−2) means the negative of '3 groups of −2'.

So we flip 3 groups of −2 as shown in Fig. 4.15 to get (−3) × (−2) = 6.

We can also write it as: −3 × (−2) = 6.

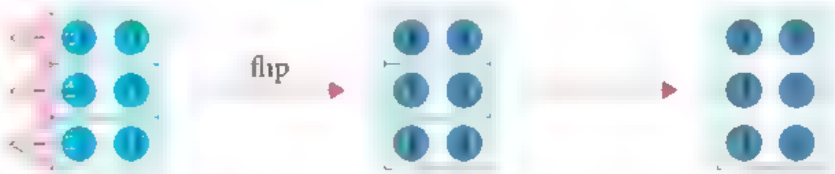


Fig. 4.15

Multiplication involving negative integers

Work out the answers to parts (a) to (d) below, using number discs as shown above. Write down the final answer for each part.

(a) 3 × 4 (i.e. 3 groups of 4) **(b)** 3 × (−4) **(c)** (−4) × 3 **(d)** (−4) × (−3)

Using the results from the above Investigation, complete the following. In general,

positive number × *positive* number = *positive* number,
positive number × *negative* number = *negative* number,
negative number × *positive* number = ______ number,
negative number × *negative* number = ______ number.

Practice Now 7A

Without using a calculator, find the value of each of the following.

(a) $2 \times (-9)$ (b) $(-8) \times 4$ (c) $(-6)(-7)$
(d) $-(-10)$ (e) $11 \times (-3)$ (f) -19×10
(g) $-12 \times (-3)$ (h) $-4(-7)$ (i) $2020 \times (-1)$

Attention

The following are different notations used to represent the same operation.
(c) $(-6)(-7)$ means $(-6) \times (-7)$.
(h) $-4(-7)$ means $-4 \times (-7)$.

C. Division involving negative integers

We have learnt that $6 \div 2 = \frac{6}{2} = 6 \times \frac{1}{2} = 3$.

Similarly, $(-6) \div 2 = \frac{-6}{2} = -6 \times \frac{1}{2} = -\left(6 \times \frac{1}{2}\right) = -3$,

$6 \div (-2) = \frac{6}{-2} = 6 \times \frac{1}{-2} = 6 \times \left(-\frac{1}{2}\right) = -3$,

$(-6) \div (-2) = \frac{-6}{-2} = -6 \times \frac{1}{-2} = -6 \times \left(-\frac{1}{2}\right) = 3$.

Hence, complete the following. In general,

positive number ÷ positive number = positive number,
positive number ÷ negative number = negative number,
negative number ÷ positive number = ______ number,
negative number ÷ negative number = ______ number.

Practice Now

Without using a calculator, evaluate each of the following.

(a) $(-8) \div 2$ (b) $15 \div (-3)$ (c) $-21 \div (-7)$
(d) $\frac{-16}{4}$ (e) $\frac{20}{-5}$ (f) $\frac{-24}{-3}$

D. Factors revisited

In Chapter 1, we have learnt about the positive factors of a non-zero whole number.

In this chapter, we have learnt that numbers can also be negative. Therefore, positive and negative integers can have either positive or negative factors as well.

Attention

We only deal with positive integers and prime factors when we carry out prime factorisation.

Worked Example 7

Positive and negative factors

The positive and negative factors of 4 are 1, −1, 2, −2, 4 and −4 (and we write ±1, ±2 and ±4 in short).
Find the positive and negative factors of 6.

*Solution

$6 = 1 \times 6$
$= 2 \times 3$
∴ positive and negative factors of 6 are ±1, ±2, ±3 and ±6.

The notation ± is used to represent '+' (positive) or '−' (negative) in a *concise manner*. For example, ±1 means +1 or −1.

1. Find the positive and negative factors of the following numbers.
   (a) 8 (b) -9 (c) 7 (d) -1

2. Write down a positive and a negative multiple of the following numbers.
   (a) 2 (b) -6 (c) 1 (d) -3

E. Square roots and cube roots revisited

In Chapter 1, we have learnt that $5^2 = 5 \times 5 = 25$ and so $\sqrt{25} = 5$. Now, what is $(-5)^2$ or $(-5) \times (-5)$ equal to?

Since $5 \times 5 = 25$ and $(-5) \times (-5) = 25$, then 25 has *two square roots*:

(i) the positive square root of 25, written as $\sqrt{25} = 5$, and

(ii) the negative square root of 25, written as $-\sqrt{25} = -5$.

The square root sign $\sqrt{\ }$ is used to denote the *positive square root* only.

We can combine both the positive and negative square roots by writing $\pm\sqrt{25} = \pm 5$.

Worked Example 8

Finding the square roots of a number

Without using a calculator, find the square roots of 49.

Solution

Square roots of $49 = \pm\sqrt{49}$
$= \pm 7$

Exercise 4C

Without using a calculator, find

(a) the square roots of 64, (b) the negative square root of 9, (c) $\sqrt{36}$.

Is it possible to obtain the square roots of a negative number, e.g. $\pm\sqrt{-16}$? Explain your answer.

In Chapter 1, we have learnt that $5^3 = 5 \times 5 \times 5 = 125$ and so $\sqrt[3]{125} = 5$.
Now, what is $(-5)^3$ or $(-5) \times (-5) \times (-5)$ equal to?
Since $(-5)^3 = -125$, unlike square roots, a number has *only one cube root*, and it is *possible* to obtain the cube root of a negative number, e.g.

$$\sqrt[3]{125} = \sqrt[3]{5\times5\times5} = 5 \text{ and } \sqrt[3]{-125} = \sqrt[3]{(-5)\times(-5)\times(-5)} = -5.$$

Further Questions
Exercise 4C
Questions 7(a)–(h)

Without using a calculator, evaluate each of the following.

(a) $(-3)^3$ (b) $\sqrt[3]{64}$ (c) $\sqrt[3]{-8}$ (d) $-\sqrt[3]{-27}$

F. Combined operations on integers

We have learnt the **four basic operations** on numbers: +, −, ×, ÷, and square, square root, cube and cube root. Let us learn how to perform all these operations together.

Operations are performed in the following order:

1. **Brackets:** Evaluate expressions in brackets first. If there are more than one pair of brackets, evaluate the expression in the innermost pair of brackets first.
2. **Powers and roots:** Evaluate the powers and roots.
3. **Multiplication and division:** Multiply and divide from the left to the right.
4. **Addition and subtraction:** Add and subtract from the left to the right.

Combined operations on integers

Without using a calculator, evaluate the following.

(a) $6 - 7 + 2 \times (4 - 3^2)$ (b) $(-2)^3 - 12 \div \left[2 - \left(\sqrt{25} + 3\right)\right]$

Solution

(a)
$$\begin{aligned} 6 - 7 + 2 \times (4 - 3^2) &= 6 - 7 + 2 \times (4 - 9) && \text{powers inside brackets} \\ &= 6 - 7 + 2 \times (-5) && \text{brackets} \\ &= 6 - 7 + (-10) && \text{multiplication} \\ &= -1 + (-10) && \text{recall: 2} \\ &= -11 && \text{recall (−2)} \end{aligned}$$

(b)
$$\begin{aligned} (-2)^3 - 12 \div \left[2 - \left(\sqrt{25} + 3\right)\right] &= -8 - 12 \div [2 - (5 + 3)] \\ &= -8 - 12 \div (2 - 8) && \text{innermost brackets} \\ &= -8 - 12 \div (-6) && \text{bracket} \\ &= -8 - (-2) && \text{divisio} \\ &= -8 + 2 && \text{recall 5 -} \\ &= -6 && \text{recall 2} \end{aligned}$$

Without using a calculator, find the value of each of the following.

(a) $-2 \times (15 - \sqrt{49} + 2^3)$ (b) $4^3 - \left\{7 \times \left[16 - \left(\sqrt[3]{64} - 5\right)\right]\right\}$

G. Using calculator to evaluate negative integers

Most calculators distinguish between 'minus' and 'negative'.

The 'minus' button is [−] but the 'negative' button is either [+/−] or [(−)] depending on the model of the calculator.

For example, to evaluate $2 - (-5)$ using a calculator, press:

[2] [−] [(] [(−)] [5] [)] [=]

For some calculators, it is not necessary to key in the brackets.

The answer to Worked Example 9 part **(b)** can be evaluated using a calculator. Press:

[(] [(−)] [2] [)] [x^3] [−] [1] [2] [÷] [(] [2] [−] [(] [∛] [2] [5] [▶] [+] [3] [)] [)] [=]

Use a calculator to check your answers in Practise Now 9A.

1. What do I already know about the multiplication and division of positive integers that could guide my learning of the multiplication and division of negative integers?
2. When evaluating a numerical expression involving +, −, ×, ÷, powers, roots and/or brackets, do I know the order in which the operations are performed?

Basic | Intermediate

Exercise

Do not use a calculator for this exercise, unless otherwise stated.

1. Evaluate the following.

(a) $5 \times (-7)$ **(b)** -8×3
(c) $(-9)(-4)$ **(d)** $-(-716)$
(e) $-1 \times (-697)$ **(f)** $-11(-8)$
(g) -6×0 **(h)** $42(-2)$

2. Find the value of each of the following.

(a) $-12 \div 4$ **(b)** $16 \div (-2)$
(c) $(-18) \div (-9)$ **(d)** $\frac{-14}{7}$
(e) $\frac{45}{-5}$ **(f)** $\frac{-18}{-3}$

3. Find the positive and negative factors of each of the following numbers.

(a) 12 **(b)** −23 **(c)** 16 **(d)** 1

4. OPEN Write down a positive and a negative multiple of each of the following numbers.
If it is not possible to do so, explain why.

(a) 5 **(b)** −8 **(c)** −17 **(d)** 0

Exercise 4C

5. Find the square roots of each of the following numbers.
(a) 81 (b) 16 (c) 25 (d) 100

6. Evaluate each of the following where possible. If it is not possible, explain why.
(a) $\sqrt{81}$ (b) $\sqrt{4}$ (c) $-\sqrt{9}$ (d) $\sqrt{-64}$

7. Find the value of each of the following where possible. If it is not possible, explain why.
(a) $(-4)^3$
(b) -4^3
(c) $-(-4^3)$
(d) $\sqrt[3]{8}$
(e) $-\sqrt[3]{125}$
(f) $\sqrt[3]{-216}$
(g) $-\sqrt[3]{-64}$
(h) $\sqrt[3]{-1000}$

8. Calculate the value of each of the following.
(a) $-55 + (-10) - 12$
(b) $-12 - [(-8) - (-2)] + 3$
(c) $-100 + (-45) + (-5) + 20$
(d) $-2 + 3 \times 15$
(e) $(-5 - 2)(-3)$
(f) $-25 \times (-4) \div (-12 + 32)$
(g) $3 \times (-3)^2 - (7 - 2)^2$
(h) $5[3 \times (-2) - 10]$
(i) $-12 + [2^2 - (-2)]$
(j) $\sqrt{10 - 3 \times (-2)}$

9. Use a calculator to check your answers to Question 8.

10. Without calculating the value of $-987 \times (-654) \div (-321)$, explain whether the value is positive or negative.

11. Write down a positive and a negative factor of 0.

OPEN

12. Is $\sqrt{n^2} = n$ always true for any integer n? Explain.

13. If n and $\frac{198}{n^2}$ are integers, find all the possible values of n.

14. Evaluate the following.
(a) $24 \times (-2) \times 5 \div (-6)$
(b) $4 \times 10 - 13 \times (-5)$
(c) $160 \div \sqrt[3]{-8} - 20 \div (-5)$
(d) $\sqrt{5^2 - 3^2} - (57 - 77) + 2$
(e) $[(12 - 18) \div 3 - (-1)] \times (-4)^3$
(f) $(5 - 2)^3 \times 2 + [-4 + (-7)] \div (-2 + 4)^2$
(g) $\{-10 - [12 + (-3)^2] + 3^3\} + (-3)$
(h) $\sqrt[3]{-2 \times (-37) - [-2(-3) + 8 \times (-2) - 8 \times 2] + 5^2}$

15. In a mathematics Olympiad, a correct answer is awarded 3 marks, but a wrong answer is awarded −1 mark (i.e. 1 mark will be deducted). If no answer is given, 0 marks will be awarded. Albert took part in this Olympiad. There were 30 questions. He answered 17 questions correctly and 5 questions wrongly. How many marks did he score?

16. Use a calculator to check your answers to Question 14.

17. A six-sided die is rolled. If it shows a prime number that is odd, 5 points are awarded. If it shows a prime number that is even, 9 points will be deducted. If it shows any other number, 0 points will be awarded. The table shows how many times each number is obtained when Shaha rolls the die 20 times.

Number shown on die	1	2	3	4	5	6
Number of times	3	4	7	0	1	5

What is Shaha's final score?

Basic Intermediate

Exercise 4C

There are 20 questions in a mathematics competition.
A correct answer is awarded 2 marks, but a wrong answer is awarded −1 mark (i.e. 1 mark will be deducted). If no answer is given, 0 marks will be awarded.

(i) Nadia answered a total of 14 questions and scored 10 marks. How many questions did she answer correctly?

OPEN (ii) Waseem scored −8 marks for the competition. He has forgotten how many questions he answered. Give an example of the number of correct answers and the number of wrong answers that he might have obtained.

Ali is learning how to code. When he enters a positive integer h into the computer program, the avatar moves forward by h units. When he enters a negative integer k into the computer program, the avatar moves backward by $-k$ units. Given that Ali enters the value h followed by the value k into the computer program, what is the total distance moved by the avatar?

If n is an integer such that $(n - 7) \times (n + 3)$ is a prime number, find all the possible values of n.

4.4 Negative fractions and mixed numbers

In Chapter 2, we have learnt about **proper fractions** (e.g. $\frac{3}{4}$), **improper fractions** (e.g. $\frac{5}{3}$ and $\frac{2}{2}$) and **mixed numbers** (e.g. $5\frac{1}{4}$).
These numbers are positive but they can be extended to include negative fractions (e.g. $-\frac{3}{4}$), and negative mixed numbers (e.g. $-5\frac{1}{4}$).
In this section, we will learn how to perform the four operations on negative fractions and mixed numbers.

Attention

Although $\frac{-3}{4} = \frac{3}{-4} = -\frac{3}{4}$, we usually write it as $-\frac{3}{4}$.

A. Basic operations on fractions and mixed numbers

In Sections 4.2 and 4.3, we have learnt how to add, subtract, multiply and divide positive and negative integers. These apply to fractions and mixed numbers as well.

Worked Example 10

Adding and subtracting negative fractions and mixed numbers

Without using a calculator, evaluate the following.

(a) $6\frac{1}{5}+\left(-2\frac{3}{10}\right)$ (b) $-\frac{7}{4}-\frac{5}{6}-\left(-1\frac{1}{3}\right)$

Solution

(a) $6\frac{1}{5}+\left(-2\frac{3}{10}\right)$

$= 6\frac{1}{5}-2\frac{3}{10}$	recall: 5 + (−2) = 5 − 2
$= 4\frac{2}{10}-\frac{3}{10}$	convert to equivalent fractions LCM of 5 and 10 is 10
$= \left(3+\frac{10}{10}\right)+\frac{2}{10}-\frac{3}{10}$	
$= 3\frac{9}{10}$	leave answer as a mixed number

(b) $-\frac{7}{4}-\frac{5}{6}-\left(-1\frac{1}{3}\right)$

$= -\frac{7}{4}-\frac{5}{6}+\frac{4}{3}$	convert to improper fraction & recall: 5 − (−2) = 5 + 2
$= -\frac{21}{12}-\frac{10}{12}+\frac{16}{12}$	convert to equivalent fractions: LCM of 3, 4 and 6 is 12
$= \frac{-21-10+16}{12}$	
$= \frac{-31+16}{12}$	recall: −5 − 2 = −7
$= \frac{16-31}{12}$	recall: −2 + 5 = 5 − 2
$= \frac{-15}{12}$	recall: 2 − 5 = −
$= -\frac{5}{4}$	reduce to simplest form
$= -1\frac{1}{4}$	leave answer as a mixed number

For addition and subtraction, always convert unlike fractions to

Similar and Further Questions
Exercise 4D

Without using a calculator, find the value of each of the following.

(a) $7\frac{1}{2}+\left(-3\frac{3}{5}\right)$ **(b)** $-3\frac{1}{3}+\left(-\frac{1}{4}\right)$

(c) $-2\frac{3}{4}-\frac{5}{6}-\left(-\frac{8}{3}\right)$ **(d)** $\frac{5}{9}-2\frac{1}{6}-\left(-\frac{4}{3}\right)$

11

Multiplying and dividing negative fractions and mixed numbers

Without using a calculator, evaluate the following.

(a) $-1\frac{5}{12}\times\left(-2\frac{7}{34}\right)$ **(b)** $3\frac{2}{3}\div\left(-2\frac{4}{9}\right)$

Solution

(a) $-1\frac{5}{12} \times \left(-2\frac{7}{34}\right) = -\frac{\cancel{17}^{1}}{12} \times \left(-\frac{75}{\cancel{34}}\right)$ convert to improper fractions

$= \frac{25}{8}$ negative number × negative number = positive number

$= 3\frac{1}{8}$ leave answer as a mixed number

(b) $3\frac{2}{3} \div \left(-2\frac{4}{9}\right) = \frac{11}{3} \div \left(-\frac{22}{9}\right)$ convert to improper fractions

$= \frac{\cancel{11}}{\cancel{3}} \times \left(-\frac{\cancel{9}}{\cancel{22}}\right)$ dividing by a fraction is equivalent to multiplying by its reciprocal

$= -\frac{3}{2}$ positive number × negative number = negative number

$= -1\frac{1}{2}$ leave answer as a mixed number

Without using a calculator, evaluate the following.

(a) $-2\frac{2}{7} \times \frac{7}{32}$

(b) $\left(-\frac{11}{19}\right) \times \left(-1\frac{7}{22}\right)$

(c) $5\frac{1}{4} \div \left(-2\frac{4}{5}\right)$

(d) $\frac{20}{7} \div \left(-1\frac{4}{21}\right)$

B. Combined operations on fractions and mixed numbers

In Section 4.3F, we have learnt that operations are performed in a particular order. Let us now apply this to fractions and mixed numbers.

Worked

12

Combined operations on negative fractions and mixed numbers

Without using a calculator, find the value of $-2\frac{4}{5} \times \left[-\frac{5}{2} + \left(-1\frac{1}{2}\right)^2\right]$.

Solution

$-2\frac{4}{5} \times \left[-\frac{5}{2} + \left(-1\frac{1}{2}\right)^2\right]$

$= -\frac{14}{5} \times \left[-\frac{5}{2} + \left(-\frac{3}{2}\right)^2\right]$

$= -\frac{14}{5} \times \left[-\frac{5}{2} + \left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right)\right]$

$= -\frac{14}{5} \times \left(-\frac{5}{2} + \frac{9}{4}\right)$ negative number × negative number = positive number

$= -\frac{14}{5} \times \left(-\frac{10}{4} + \frac{9}{4}\right)$ convert to equivalent fractions

$= -\frac{14}{5} \times \left(\frac{9}{4} - \frac{10}{4}\right)$ recall −2 + 5 = 5 − 2

$= -\frac{\cancel{14}^{7}}{5} \times \left(-\frac{1}{\cancel{4}_{2}}\right)$ recall 2 − 5 = −3

$= \frac{7}{10}$ negative number × negative number = positive number

Practise Now 12A

Similar and Further Questions

Exercise 4D

Without using a calculator, evaluate the following.

(a) $-2\frac{2}{11} \div \left[-\frac{18}{55}\times\left(-\frac{5}{6}\right)\right]$

(b) $-\frac{4}{7}\times\left[\left(\frac{1}{2}\right)^2-\left(-1\frac{3}{4}\right)\right]$

(c) $\left[-\frac{3}{4}+\left(-\frac{5}{2}\right)\right]\div\frac{2}{13}$

(d) $-\frac{9}{14}\times\left[\left(-\frac{3}{2}\right)^2-2\frac{5}{6}\right]$

C. Using calculator to evaluate fractions and mixed numbers

We have learnt in Chapter 1 that the 'fraction' button on most calculators is $\boxed{\frac{\square}{\square}}$.

To key in a mixed number, we press: $\boxed{\text{SHIFT}}$ $\boxed{\frac{\square}{\square}}$.

For example, to key in $3\frac{4}{5}$, press $\boxed{\text{SHIFT}}$ $\boxed{\frac{\square}{\square}}$ $\boxed{3}$ $\boxed{\blacktriangleright}$ $\boxed{4}$ $\boxed{\blacktriangledown}$ $\boxed{5}$

To convert a mixed number to an improper fraction and vice versa, press: $\boxed{\text{SHIFT}}$ $\boxed{\text{S}\Leftrightarrow\text{D}}$.

Practise Now 12B

Similar and Further Questions

Exercise 4D

Use a calculator to check your answers in Practise Now 10, 11 and 12A.

4.5 Negative decimals

We have learnt about decimals such as 0.716 and 7.14.
These decimals are positive but they can be extended to include negative decimals (e.g. –0.716).
In this section, we will learn how to perform the four operations involving negative decimals.

A. Basic operations on negative decimals

In Sections 4.2 and 4.3, we have learnt to perform basic arithmetic operations involving positive and negative integers, which apply to fractions and mixed numbers as we learnt in Section 4.4.

In fact, all these also apply to decimals.

Worked Example 13

Adding and subtracting decimals

Without using a calculator, evaluate the following.

(a) −97.5 − 8.743 **(b)** 7.2 − 17

Solution

(a)

$$\begin{array}{rl} {}^{1}9\,{}^{1}7.5\,0\,0 & \leftarrow \text{put two zeros here so that the two numbers have the same number of decimal places} \\ +\quad 8.7\,4\,3 & \\ \hline 1\,0\,6.2\,4\,3 & \end{array}$$

∴ −97.5 − 8.743 = −106.243 recall

(b) align decimal points

$$\begin{array}{rl} {}^{0}\cancel{1}\,{}^{6}\cancel{7}.{}^{1}0 & \leftarrow \text{put a zero here so that the two numbers have the same number of decimal places} \\ -\quad 7.2 & \\ \hline 9.8 & \end{array}$$

∴ 7.2 − 17 = −9.8 recall 2

When adding or subtracting decimals, we align the decimal points of the numbers.

Practice Now 13

Without using a calculator, find the value of each of the following.

(a) −93.8 − 7.236 **(b)** 5.4 − 15 **(c)** 124.8 + (−7.24) − (−22.44)

Worked Example 14

Multiplying and dividing negative decimals

Without using a calculator, evaluate the following.

(a) −14.52 × 6.8 **(b)** −6.45 ÷ (−1.2)

Solution

(a) −14.52 × 6.8 = −98.736 negative number × positive number = negative number

$$\begin{array}{r} 1\,4.5\,2 \\ \times \quad 6.8 \\ \hline 1\,1\,6\,1\,6 \\ +\ 8\,7\,1\,2\quad \\ \hline 9\,8.7\,3\,6 \end{array}$$

(b) $-6.45 \div (-1.2)$
$= -64.5 \div (-12)$
$= 5.375$ negative number ÷ negative number = positive number

$$
\begin{array}{r}
5.375 \\
12\overline{)\,64.500} \\
-\ 60 \\
\hline
4\ 5 \\
-\ 3\ 6 \\
\hline
9\ 0 \\
-\ 8\ 4 \\
\hline
6\ 0 \\
-\ 6\ 0 \\
\hline
0
\end{array}
$$

Similar and

Exercise 2D

Questions 5(a)–(f)

Without using a calculator, find the value of each of the following.

(a) $0.19 \times (-4.5)$ **(b)** $-5.64 \times (-0.678)$

(c) $42 \div (-1.6)$ **(d)** $-120.254 \div 2.5$

B. Combined operations on decimals

In Section 4.3, we have learnt how to perform combined operations involving negative integers, which was extended to fractions and mixed numbers in Section 4.4.

Let us apply what we have learnt to decimals.

Worked Example 15

Combined operations on negative decimals

Without using a calculator, evaluate the following.

(a) $\frac{0.18}{0.3} \times \left(\frac{-0.47}{1.2}\right)$ **(b)** $-2.3 \times [-5.7 + (-2.31)]$

Solution

(a) $\frac{0.18}{0.3} \times \left(\frac{-0.47}{1.2}\right) = \frac{1.8}{3} \times \left(\frac{-0.47}{1.2}\right)$

$= 0.6 \times \left(\frac{-0.47}{1.2}\right)$

$= \cancel{6} \times \frac{-0.47}{\cancel{12}_{2}}$

$= -0.235$ positive number × negative number = negative number

(b) $-2.3 \times [-5.7 + (-2.31)]$
$= -2.3 \times (-8.01)$ recall: $(-2) + (-3) = -5$
$= 18.423$ negative number × negative number = positive number

Attention

(a)
$$
\begin{array}{r}
0.6 \\
3\overline{)\,1.8} \\
-1\ 8 \\
\hline
0
\end{array}
\qquad
\begin{array}{r}
0.2\ 3\ 5 \\
2\overline{)\,0.4\ 7\ 0} \\
-\ 4 \\
\hline
7 \\
-\ 6 \\
\hline
1\ 0 \\
-\ 1\ 0 \\
\hline
0
\end{array}
$$

(b)
$$
\begin{array}{r}
8.0\ 1 \\
\times\quad 2.3 \\
\hline
2\ 4\ 0\ 3 \\
+\ 1\ 6\ 0\ 2 \\
\hline
1\ 8.4\ 2\ 3
\end{array}
$$

Without using a calculator, find the value of each of the following.

(a) $\frac{0.12}{0.4} \times \left(\frac{-0.23}{0.6}\right)$ (b) $-7.2 \times [-1.3 + (-3.5)]$

(c) $-0.3^2 \times 4.5 \div (-2.7) - 0.65$ (d) $3.2^3 \times (-0.625)^2 \div (-6.4)$

4.6 Rational, irrational and real numbers

A. Rational and irrational numbers

Whole numbers, integers and fractions are all different types of numbers. Some of these numbers are *rational numbers* while others are not.

A **rational number** is a number that can be expressed as the **ratio** of two integers a and b, i.e. in the form $\frac{a}{b}$, where $b \neq 0$

Examples of rational numbers are 2, $\frac{1}{3}$, $-5\frac{1}{8}$ and 1.6.

Numbers that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, are called **irrational numbers**, e.g. $\sqrt{7}$ and $\sqrt[3]{5}$.

Attention

The term 'fractions' is ambiguous because it can refer to *different concepts* such as:

(a) positive proper fractions (i.e. less than 1),

(b) non-integers,

(c) improper fractions, which include integers, e.g. $\frac{6}{3}$,

(d) irrational fractions, e.g. $\frac{\sqrt{7}}{2}$.

∴ we *cannot* say that rational numbers are fractions, and vice versa.

Rational and irrational numbers

1. Are integers (such as 2 and −3) rational numbers? Explain.
2. Are all square roots and cube roots irrational numbers? Explain.
3. The number π is a mathematical constant that is used when calculating the circumference and area of a circle (see Chapter 12). It is sometimes taken to be $\frac{22}{7}$.
   Is π a rational number? To answer this question, use a calculator to
   (a) evaluate $\frac{22}{7}$ and write down its value,
   (b) find the value of π by pressing SHIFT π.
   Compare their values. Why are they different?

Attention

The π value may be obtained from different keys, depending on the model of your calculator. If there is no π key, look for the key that π appears above of. Press SHIFT followed by this key.

B. Real numbers

Real numbers are made up of rational numbers and irrational numbers.
Fig. 4.16 illustrates the relationships among the different types of numbers.
Can you give some other examples for each category of numbers?

- Real numbers
  - Rational numbers
    - Integers
      - Whole numbers
        - Positive integers (e.g. 2, 8)
        - Zero
      - Negative integers (e.g. −4, −7)
    - Non-integer rational numbers (e.g. $\frac{1}{3}$, $-5\frac{1}{8}$)
  - Irrational numbers (e.g. π, $\sqrt{7}$, $-\sqrt[3]{5}$)

Fig. 4.16

C. Decimal representations of rational, irrational and real numbers

Terminating, recurring and non-recurring decimals

Copy and complete Table 4.1 by using a calculator to evaluate each of the following. Write down all the numbers displayed on the calculator.

Group 1	Group 2	Group 3
$\frac{7}{4} =$	$\frac{1}{3} =$	$\sqrt{2} =$
$-3\frac{1}{8} =$	$-\frac{123}{99} =$	$-\sqrt[3]{5} =$
$\frac{63}{64} =$	$\frac{22}{7} =$	$\pi =$

Table 4.1

1. (i) Are the numbers in Group 1 rational or irrational numbers? Explain.
   (ii) Are the *decimal representations* (i.e. the calculator values) of the numbers in Group 1 terminating?
2. (i) Are the numbers in Group 2 rational or irrational numbers? Explain.
   (ii) What do you notice about the decimal representations of the numbers in Group 2?
   Do the decimals terminate, or do they recur?
   (The last digit of $\frac{22}{7}$ in the calculator display may have been rounded up, so you may not see the pattern.
   The actual value of $\frac{22}{7}$ is 3.142 857 142 857 142 857...)
3. (i) Are the numbers in Group 3 rational or irrational numbers? Explain.
   (ii) What do you notice about the decimal representations of the numbers in Group 3? Do the decimals terminate, recur, or neither?
4. (i) From your answers to Questions 1 to 3, how do you tell whether a number is rational or irrational by its decimal representation?

(ii) Since real numbers are made up of rational and irrational numbers, what can you say about the decimal representations of real numbers (in terms of terminating and/or recurring)?

Exercise 4D
Question

5. (i) How can you tell if $\frac{7}{4}$ or $\sqrt{2}$ is larger?

Decimal representations are useful to compare non-integer real numbers.

Arrange the numbers in Table 4.1 in ascending order, i.e. from the smallest to the greatest.

(ii) We can also use a dot to represent a real number, such as $\frac{1}{3}$, on a number line (see Fig. 4.17).

Use a dot to represent each of the numbers in Table 4.1 on the number line below

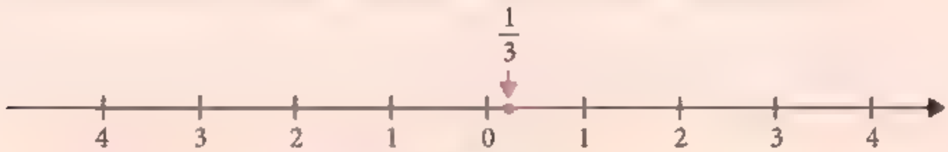


Fig. 4.17

From the above Investigation, we observe that:

Real numbers are numbers that can be represented by values on a number line. They can be written as decimal representations, which can be classified into three types:

- **Terminating decimals**, i.e. the digits after the decimal point terminate.
- **Recurring (or repeating) decimals**, i.e. some digits after the decimal point repeat themselves indefinitely.
- **Non-terminating and non-recurring decimals**, i.e. the digits after the decimal point do not repeat but they continue indefinitely.

Rational numbers are terminating decimals or recurring decimals.
Irrational numbers are non-terminating and non-recurring decimals.

...tions and Equivalence

Real numbers can be represented in different notations that are equivalent, e.g. $\frac{7}{4} = 1.75$,

$\frac{1}{3} = 0.333...$ (or $0.\dot{3}$) and

$\sqrt{2} = 1.14142...$

Fig. 4.18 illustrates the relationships among the different types of numbers

Real numbers

Rational numbers | Irrational numbers

Terminating decimals (e.g. $2, -0.7, \frac{9}{4}, -3\frac{1}{8}$)

Recurring decimals (e.g. $0.\dot{3}, -\frac{123}{99}, \frac{22}{7}$)

Non-terminating and non-recurring decimals (e.g. $\sqrt{2}, -\sqrt[3]{5}, \pi$)

Fig. 4.18

16

Using calculator to evaluate real numbers

Use a calculator to evaluate $\dfrac{3.2\pi+4.3^2}{\sqrt{47.5-2\frac{3}{4}}}$, leaving your answer correct to 3 decimal places.

Solution

Sequence of calculator keys:

(3 . 2 SHIFT π + 4 . 3 x^2)
÷ (√ 4 7 . 5 ▷ − SHIFT ⊟
2 ▷ 3 ▽ 4 ▷) =

$\dfrac{3.2\pi+4.3^2}{\sqrt{47.5-2\frac{3}{4}}} = 6.891$ (to 3 d.p.)

Attention

We can also use another method: press the ⊟ button first and key in the respective numerator and denominator.

Practise Now 16

Further Questions
Exercise 4D
Question 13(a)–(d)

Use a calculator to evaluate $\dfrac{\pi\times0.7^2}{\sqrt[3]{2.4+1\frac{3}{10}}}$, leaving your answer correct to 3 decimal places.

1. How did what I learnt about the four operations (+, −, ×, ÷) on positive and negative integers in Sections 4.2-4.3 help me in performing the four operations on fractions, mixed numbers and decimals in Sections 4.4-4.5?
2. What have I learnt in this section or chapter that I am still unclear of?

Advanced | Intermediate | Basic

Exercise 4D

Do not use a calculator for this exercise, unless otherwise stated.

1. Find the value of each of the following.
   (a) $-\frac{1}{2}+\left(-\frac{3}{4}\right)$ (b) $3\frac{1}{5}-5\frac{1}{2}$
   (c) $3\frac{1}{8}+\left(-\frac{1}{4}\right)$ (d) $4\frac{1}{6}-\left(-4\frac{2}{3}\right)$
2. Evaluate the following.
   (a) $\frac{64}{15}\times\left(-\frac{3}{8}\right)$ (b) $\frac{4}{15}\div\left(-\frac{10}{3}\right)$
   (c) $-6\frac{1}{8}\times\frac{3}{14}$ (d) $-2\frac{1}{2}\times4\frac{2}{5}$
   (e) $-1\frac{1}{4}\div\frac{3}{8}$ (f) $-\frac{8}{9}\div\left(-1\frac{2}{3}\right)$
3. Use a calculator to check your answers to
   (a) Question 1, (b) Question 2.

Exercise

4. Find the value of each of the following.
   (a) $(-4.6)+(-3.2)$ (b) $2.5-6.8$
   (c) $6.9+(-4.7)$ (d) $12.6+(-35.7)$
   (e) $-29.3+11.2$ (f) $-15.3-27.9$
   (g) $11.8-(-2\ 3)$ (h) $-0.7-(-1.5)$

5. Evaluate each of the following.
   (a) 14.72×1.3 (b) $(-4.6)\times(-0.1833)$
   (c) $2\ 35\times(-0.52)$ (d) $3.426\div(-0.06)$
   (e) $-1.32\div 0.12$ (f) $-0.16\div 0.125$

6. Evaluate the following.
   (a) $4\frac{2}{7}-6\frac{1}{3}-\left(-\frac{8}{21}\right)$
   (b) $2\frac{1}{5}-\left(-\frac{3}{4}\right)+\left(-7\frac{1}{10}\right)$
   (c) $-9+\left(-2\frac{1}{8}\right)+\left(-\frac{4}{3}\right)$
   (d) $-\frac{1}{5}+\frac{19}{4}+\left(-\frac{7}{2}\right)$
   (e) $-\frac{8}{3}-\frac{1}{6}-\left(-2\frac{1}{8}\right)$
   (f) $5\frac{3}{4}-2\frac{5}{6}+\left(-\frac{23}{15}\right)-\left(-4\frac{7}{10}\right)$

7. Calculate the value of each of the following.
   (a) $-\frac{5}{7}\times\left(-\frac{28}{15}+1\frac{2}{3}\right)$
   (b) $\left[-\left(\frac{1}{2}\right)^2-\left(-\frac{1}{3}\right)\right]\div\left(\frac{1}{4}-\frac{1}{3}\right)$
   (c) $10-\frac{15}{8}\times\left(\frac{3}{2}+4\frac{1}{2}\right)+\left(-\frac{1}{4}\right)$
   (d) $\left(-\frac{3}{2}\right)^2\times\left(\frac{1}{15}-2\frac{1}{3}\right)$

8. Use a calculator to check your answers to
   (a) Question 6, (b) Question 7.

9. Use a calculator to evaluate the following.
   (a) $\left(\frac{1}{2}\right)^3-\left(2\frac{3}{4}\right)^2+\left(-\frac{3}{4}\right)$
   (b) $\frac{7}{3}+\frac{4}{9}\times\left(-\frac{1}{2}\right)^2$
   (c) $-\frac{11}{4}-\sqrt[3]{2-5\frac{3}{8}}$
   (d) $\sqrt{\frac{11}{12}\div\left(\frac{5}{2}-\frac{5}{6}\times\frac{19}{10}\right)}$

10. Find the value of each of the following.
    (a) $-88.8-16.24$
    (b) $4.73-14$
    (c) $7.513+(-18.9)$
    (d) $-23.6+18.251$
    (e) $123.4-(-5.67)$
    (f) $-53.9-(-32.17)$
    (g) $73.8-(-5.79)+(-16.732)$
    (h) $-18.913-(-2.78)-7.8$

11. Evaluate the following.
    (a) $\frac{0.15}{0.5}\times\left(\frac{-0.16}{1.2}\right)$
    (b) $\frac{0.027}{0.03}\times\left(\frac{1.4}{-0.18}\right)$
    (c) $-0.4^2\times(\ 1.3)\div 0.8-0.62$
    (d) $(-0.2)^3\times\frac{27}{1.6}+0.105$
    (e) $-2.4-(-1.6)^2\div(-0.8)^3$
    (f) $[-7.54+(-5.79)]\times[7\div(-4)]$

12. Use a calculator to check your answers to Question 11.

Exercise

13. Use a calculator to evaluate each of the following, leaving your answer correct to 3 decimal places.

(a) $\left(\dfrac{\pi+5\frac{1}{2}}{-2.1}\right)^2$ (b) $-\dfrac{\pi^2+\sqrt{2}}{7-\sqrt[3]{4}}$

(c) $\dfrac{\sqrt[3]{14^2+19^2}}{\pi-4.55}$ (d) $\sqrt{\dfrac{4.6^2+8.3^2-\left(6\frac{1}{2}\right)^2}{2\times4.6-8.3}}$

OPEN How do we write a decimal that is non-terminating and non-recurring?

We can use a pattern that does not repeat itself. For example, the sequence of positive integers (i.e. 1, 2, 3, 4, ...) does not repeat itself.

So a decimal that is non-terminating and non-recurring is 0.123 456 789 101 112 131 415 16..., where the successive digits after the decimal point are made up of consecutive positive integers.

Write down two other decimals that are non-terminating and non-recurring.

In this chapter, we extend our understanding of the number system to include integers, rational numbers and irrational numbers. Learning to perform the four operations on this extended number system has shown us how new mathematical ideas are built upon our current knowledge. But why do we need to extend our number system?

Our number system needs to describe or quantify the world around us adequately. For example, negative numbers allow us to specify heights below sea level More importantly, there are gaps in the number system you have learnt in primary school.

Let us take a look at the following number line:

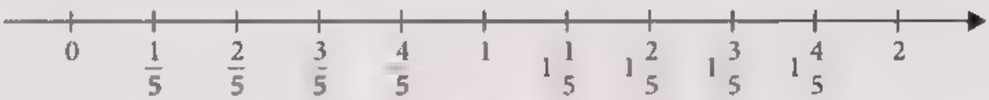


Fig. 4.19

Does it remind you of a ruler? A number line is a useful diagram for us to visualise the relative positions of the numbers we use. We can use it to pinpoint the location of a number and to measure length. However, to measure length accurately, we need a number system that can measure all lengths. Rational numbers are not enough to cover the entire number line. For instance, $\sqrt{2}$ cannot be located on the number line using only rational numbers. Thus, irrational numbers, together with rational numbers, complete the real number line. The introduction of irrational numbers shows us the usefulness of decimal representations to write real numbers, which can be represented in different notations that are equivalent (e.g. $\frac{1}{2} = 0.5$ and $\frac{1}{3} = 0.\dot{3} = 0.333...$). All these ideas set the stage for our next two questions: Is our real number system extensive enough? Are there numbers beyond the real numbers?

These questions are interesting and important as we continue to examine the world through the lens of mathematics.

Summary

1. **Real numbers** are numbers that can be represented by values on a number line. They can be classified as rational numbers and irrational numbers as shown in Fig. 4.20.
   - Give two examples of each of the following: positive integers; negative integers; non-integers; and irrational numbers.

- Real numbers
  - Rational numbers (terminating decimals or recurring decimals)
    - Integers
      - Whole numbers
        - Positive integers (e.g. 2, 8)
        - Zero
      - Negative integers (e.g. -4, -7)
    - Non-integer rational numbers (e.g. $-5\frac{1}{8}$, $0.\dot{3}$, 9.6)
  - Irrational numbers (non-terminating and non-recurring decimals) (e.g. $\frac{1}{\sqrt{2}}$, $-\sqrt[3]{5}$, π)

Fig. 4.20

2. **Addition and subtraction involving negative numbers:**

Addition of two negative numbers:	e.g. $(-2) + (-3) = -(2 + 3) = -5$
Addition of a positive and a negative number:	e.g. $5 + (-2) = 5 - 2 = 3$ and $-2 + 5 = 5 - 2 = 3$
Subtraction between two positive numbers:	e.g. $2 - 5 = -(5 - 2) = -3$
Subtraction of a positive number from a negative number:	e.g. $-5 - 2 = -(5 + 2) = -7$
Subtraction of a negative number:	e.g. $5 - (-2) = 5 + 2 = 7$

   - Give another example of each of the above operations.
   - Think of a real-life problem involving negative numbers, and solve it.

3. **Multiplication and division involving negative numbers:**

Multiply or Divide	Positive number	Negative number
Positive number	Positive number	Negative number
Negative number	Negative number	Positive number

   - Give an example of each of the above rules.
   - Think of a real-life problem that you can solve using one of the above rules.

4. **Square roots and cube roots**

   (a) A positive number (e.g. 64) has two square roots (i.e. $\pm\sqrt{64} = \pm 8$) but only one cube root (i.e. $\sqrt[3]{64} = 4$)

   (b) A negative number (e.g. -64) has no square root but has one cube root (i.e. $\sqrt[3]{-64} = -\sqrt[3]{64} = -4$).

CHAPTER 5

Approximation and Estimation

In our daily lives, there are many instances when we need to use approximations and estimations.

For example, a tourist in Singapore wanted to know how far Marina Bay Sands is from Changi Airport
"I think it is about 10 km," Vasi replied, making an estimation.
10 km is an estimated value, which can be close to, or far from, the actual value.

Albert checked online and found that the driving distance from Marina Bay Sands to Changi Airport was given to be 18.6 km.
"An online website states that the driving distance from Marina Bay Sands to Changi Airport is approximately 19 km," Albert told the tourist.
18.6 km and 19 km are approximate values, which are rounded off from the actual value, assuming that the website is correct.

In this chapter, we will learn to make appropriate approximations and estimations.

Learning Outcomes

What will we learn in this chapter?

- What approximation and estimation are
- How to round off numbers to a required number of decimal places and significant figures
- What upper and lower bounds are and how to determine them
- Why approximation and estimation have useful applications in real life

Introductory Problem

NEWS

Jinnah International Airport handles Record 7 Million Passengers

Karachi: Jinnah International Airport handled a record of 7 267 582 passengers in 2017-2018, making it one of the busiest airports in the world by passenger traffic. First opened in 1947, Jinnah International Airport currently serves over 25 airlines from around the world. The sole terminal of the airport comprises a concourse and five floors, which allows it to have an annual handling capacity of 12 million passengers.

1. Which numbers in the article are actual values, approximate values or estimated values? How can you tell?
2. Being able to distinguish between an approximate value and an estimated value helps us make better meaning of what we read. Based on your answers to Question 1, explain the difference between 'approximation' and 'estimation'.
3. **(a)** Why does the article mention 'over 25 airlines' instead of specifying the actual number of airlines?
   (b) Why does the title of the article use 7 million passengers instead of 7 267 582 passengers?

From the Introductory Problem, we have learnt that actual values, approximate values and estimated values are different.

- **Approximation** is the process of rounding off a *given* number to give an approximate value.
- **Estimation** is the process of guessing the value of an *unknown* quantity in real life. We usually give an estimated value by saying that the quantity is approximately a certain value.

In this chapter, we will learn how to round off a given number to a certain number of what we call 'significant figures', and how to estimate an unknown quantity in real life.

Let us first recap how to round off a number to the nearest 10, 100 and 1000, and to a certain number of decimal places.

5.1 Rounding and significant figures

A. Rounding off whole numbers and decimals (Recap)

We have learnt in primary school how to round off a number to the nearest 10, 100 and 1000.
We have also learnt how to round off a decimal to the nearest whole number and to a given number of decimal places.

Rounding off whole numbers and decimals

Round off each of the following numbers to the nearest 10.

(a) 275 (b) 273.1

(a) 2 7 5 = 2 8 0 (to the nearest 10)

Step 1: Identify the digit in the *tens* place.

Step 2: Examine the next digit on its right. If it is *5 or more*, we round up by adding 1 to the digit in the tens place.

Step 3: Put a zero in the ones place as a place holder.

Attention

275 is exactly midway between 270 and 280. By convention, it is *rounded up* to 280. Do not omit Step 3 because 275 = 28 (to the nearest 10) is wrong! Since the degree of accuracy is specified, we use the equal sign: 275 = 280 (to the nearest 10). If the degree of accuracy is not important, we will use the approximation sign: 275 ≈ 280.

(b) 2 7 3 . 1 = 2 7 0 (to the nearest 10)

Step 1: Identify the digit in the [illegible] place.

Step 2: Examine the next digit on its right. If it is *less than 5*, we round down and the digit in the [illegible] place remains the same.

Step 3: Put a [illegible] in the [illegible] place as a place holder. Omit all the digits after the decimal point.

Attention

273.1 is nearer to 270 than to 280, so 273.1 is *rounded down* to 270.

Practice Now

1. Round off 3 409 725 to the nearest
   (a) 10, (b) 100, (c) 1000, (d) 10 000.
2. In 2019, Singapore welcomed 19 100 000 overseas visitors. This value has been rounded off to the nearest 100 000. What is the largest and the smallest possible number of overseas visitors?

2

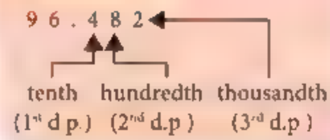
Rounding off decimals

Correct 96.482 to

(a) 1 decimal place, (b) the nearest whole number.

Attention

'Correct 96.482 to 1 decimal place' is the same as 'round off 96.482 to 1 decimal place'.



Solution

(a) 9 6 . 4 8 2 = 9 6 . 5 (to 1 d.p.)

Step 1: Identify the digit in the *tenths* place.

Step 2: Examine the next digit on its right. If it is *5 or more*, we round up by adding 1 to the digit in the tenths place.

Step 3: Do not add zero after the first decimal place.

(b) 9 6 . 4 8 2 = 9 6 (to the nearest whole number)

Step 1: Identify the digit in the *ones* place.

Step 2: Examine the next digit on its right. If it is *less than 5*, we round down and the digit in the [illegible] place remains the same.

Step 3: Do not add zero after the ones place.

Attention

For (b), do not use the answer 96.5 in (a) to round off to 97 because 96.482 is nearer to 96 than to 97. *You must not round off twice!*

Practise Now 2A

Exercise 5A

10(a)–(d), 11, 19

1. Correct 78.4695 to
   (a) one decimal place, (b) the nearest whole number,
   (c) the nearest hundredth, (d) the nearest 0.001.
2. Waseem says that 8.395 is equal to 8.4 when rounded off to 2 decimal places because he thinks that 8.40 is the same as 8.4. Do you agree? Explain your answer.

B. Significant figures

Which answer is more accurate?

1. Joyce measured a piece of string with a ruler and found its length to be 714.6 cm or 7.146 m. Raju asked Joyce to give him the length of the string, accurate to one decimal place.
   (a) Should Joyce give the answer as 714.6 cm or 7.1 m?
   (b) Which answer is more accurate? Why?
   (c) Does accuracy depend on the number of decimal places a number has? Explain.

2. Newspaper A reported that Jinnah International Airport in Pakistan handled a record of 7 267 582 passengers in 2017-2018. Newspaper B reported that Jinnah International Airport handled a record of 7 000 000 passengers in 2017-2018.
   (a) Which one of the two numbers is more accurate? Why?
   (b) Does accuracy depend on the number of digits a number has? Explain.

From the above Class Discussion, we see that the degree of accuracy of a number does not depend on **(a)** how many decimal places it has, or **(b)** how many digits it has.
Instead, the degree of accuracy depends on the number of important or *significant* digits.
For example, we say that 714.6 cm has 4 **significant figures** while 7.1 m has only 2 significant figures.

> Significant figures are used to reflect the degree of accuracy.
> A number is more accurate when it is given to a greater number of significant figures.

How many significant figures do 62 219 573 and 62 000 000 have?
It is *not* so straightforward that 62 000 000 has 2 significant figures; in fact, it can have 3 or more significant figures.
Let us learn how to identify which digits in a number are significant.

C. Five rules to identify digits which are significant

Rule 1: Non-zero digits
All non-zero digits are significant.
For example, the number 7258 has 4 significant figures.

> **Rule 1:** All [illegible] are [illegible].

[illegible]

State the number of significant figures in each of the following.
(a) 192 (b) 83.761 (c) 3 (d) 4.5

Rule 2: Zeros between non-zero digits
All zeros between non-zero digits are significant.
For example, the number 302.008 has 6 significant figures.

> **Rule 2:** All [illegible] are [illegible]

[illegible]

State the number of significant figures in each of the following.
(a) 5062 (b) 1.09 (c) 3.0024 (d) 70.8001

Rule 3: Zeros before non-zero digits in decimals

Imran measures a piece of ribbon and finds it to be 5.7 cm long.
How many significant figures does 5.7 cm have?

Imran then converts it into metres to get 0.057 m.
How many significant figures does 0.057 m have?

We see that changing the unit of measurement does not make a measurement more accurate.
Since both numbers have the same accuracy, 0.057 m has the same number of significant figures as 5.7 cm.
Thus, the zeros before the number '5' are not significant.

Rule 3: In a decimal, all [illegible] are [illegible].

[illegible]

State the number of significant figures in each of the following.
(a) 0.021 (b) 0.603 (c) 0.001 74 (d) 0.109 08

Rule 4: Zeros after non-zero digits in decimals

Li Ting and Vasi measure the volume of water in a cup using two different measuring cylinders. The readings are shown in Fig. 5.1.

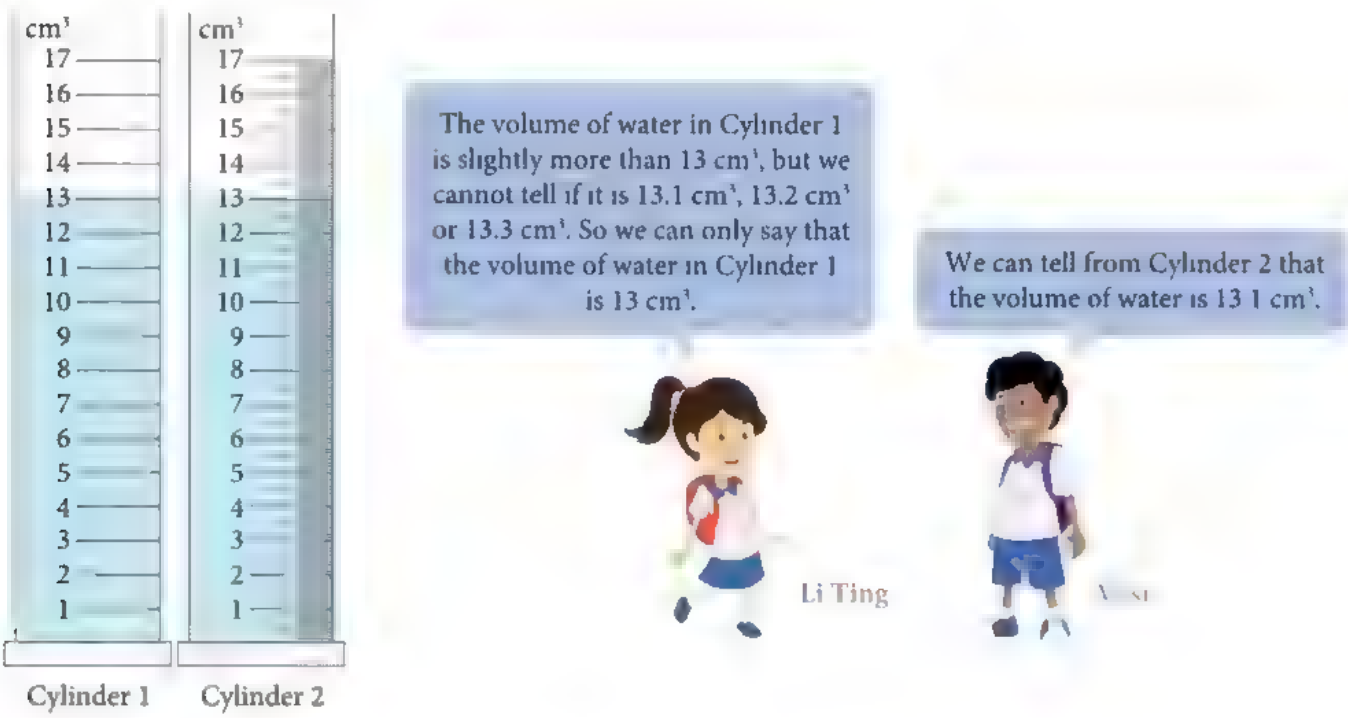


Fig. 5.1

Li Ting's reading, 13 cm³, has 2 significant figures while Vasi's reading, 13.1 cm³, has 3 significant figures.
Which reading is more accurate? Why?

Li Ting and Vasi are asked to measure the volume of water in another cup using Cylinder 1 and 2. The rea[illegible] re shown in Fig. 5.2.

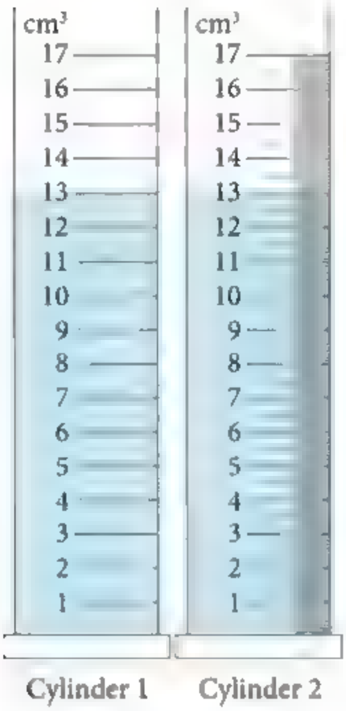
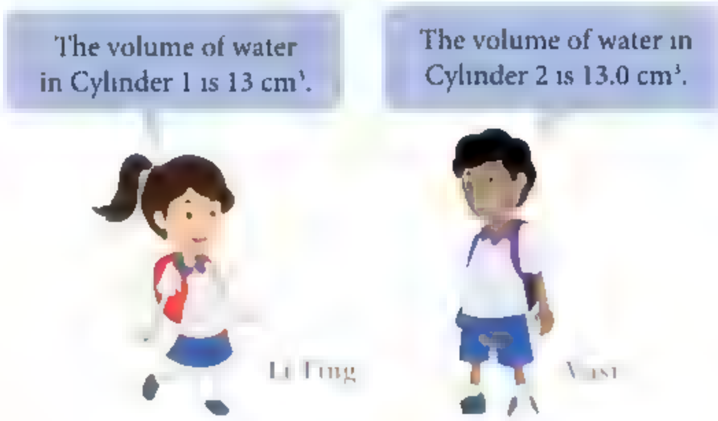


Fig. 5.2



What is the difference between 13 cm³ (for Cylinder 1) and 13.0 cm³ (for Cylinder 2)?

Cylinder 1 only measures up to 1 cm³, so we *cannot* say that the volume of water in Cylinder 1 is 13.0 cm³. Cylinder 2 measures up to 0.1 cm³, so the zero in 13.0 cm³ is meaningful and important or **significant**. It tells us that the volume of water is not 12.9 cm³ or 13.1 cm³, but 13.0 cm³.

We *cannot* say that the volume of water in Cylinder 2 is 13 cm³ because it suggests that the cylinder can only measure up to 1 cm³.

Rule 4: In a decimal, all [illegible] are [illegible].

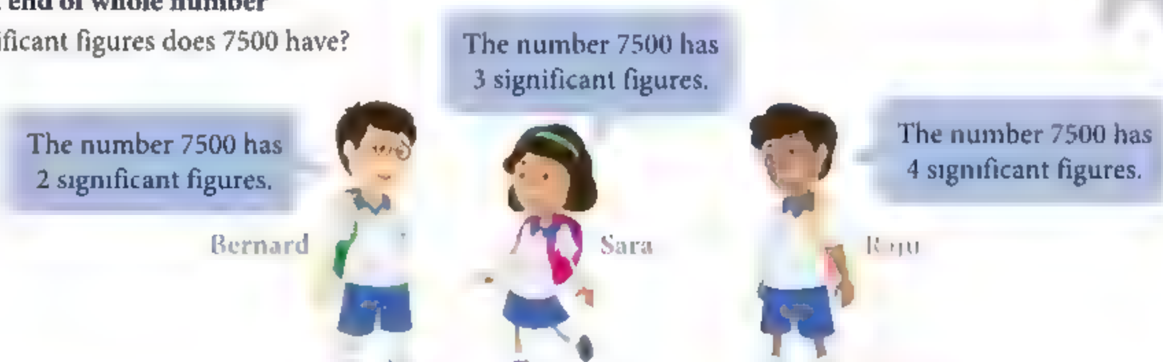
[illegible]

1. State the number of significant figures in each of the following.
   (a) 0.10 (b) 0.0500
   (c) 41.0320 (d) 6.090

2. A line segment is measured using two different instruments and its length is found to be 4.1 cm and 4.10 cm respectively. Which is more accurate and why?

Rule 5: Zeros at end of whole number

How many significant figures does 7500 have?



Which of the three students is correct?

To answer this question, let us consider the number 7498.
Round off this number to the nearest hundred (or 2 significant figures).
What do you get?
Round off this number to the nearest ten (or 3 significant figures). What do you get?
Are both the answers equal?

> **Attention**
>
> 7498, when rounded off to the nearest hundred, is *not* 75 because 75 is so far away from 7498.

Hence, Bernard and Sara are correct. 7500 has 2 significant figures when it is correct to the nearest hundred, and has 3 significant figures when it is correct to the nearest ten.

Now, what about Raju's statement? Can you think of a number that, when rounded to 4 significant figures, will give 7500?
Can 7500 have 5 significant figures?

> **Rule 5:** The [illegible] may or may not be significant, depending on how the number is rounded off.

Exercise 5A

State the number of significant figures in each of the following.

(a) 3800 m (correct to the nearest 10 m)

(b) 25 000 km (correct to the nearest km)

(c) 100 000 g (correct to the nearest 10 000 g)

D. Rounding off to given number of significant figures

Worked Example 3

Rounding off to given number of significant figures

Round off each of the following to the number of significant figures given in brackets.

(a) 8982 (2 s.f.) **(b)** 0.006 0195 (4 s.f.) **(c)** 0.9999 (3 s.f.)

(a) 8 9 8 2 = 9 0 0 0 (to 2 s.f.)

Step 1: Identify the s.f. starting from the left (2 s.f.) (Rule 1)

Step 2: Examine the next digit on its right. If it is *5 or more*, we round up by adding 1 to the previous digit.

Step 3: Put a zero to indicate the 2nd s.f. (Rule 5. This zero is significant.)

Step 4: Put two zeros as place holders. (Rule 5: These two zeros are not significant.)

(b) 0 . 0 0 6 0 1 9 5 = 0 . 0 0 6 0 2 0 (to 4 s.f.)

Not significant (Rule 3)

Step 1: Identify the s.f. starting from the left (4 s.f.) (Rules 2 and 3)

Step 2: Examine the next digit on its right. If it is *5 or more*, we round up by adding 1 to the previous digit.

Step 3: Put a zero to indicate the 4th s.f. (Rule 4)

(c) 0 . 9 9 9 9 = 1 . 0 0 (to 3 s.f.)

Not significant (Rule 3)

Step 1: Identify the s.f. starting from the left (3 s.f.). (Rule 1)

Step 2: Examine the next digit on its right. If it is *5 or more*, we round up by adding 1 to the previous digit.

Step 3: Put two zeros to indicate the 2nd and 3rd s.f. (Rule 4). Do not put a zero at the thousandths place which was previously occupied by the 3rd s.f., or else 1.000 will have 4 s.f.

1. Round off each of the following to the number of significant figures given in brackets.
   (a) 3748 (3 s.f.) **(b)** 0.004 709 89 (4 s.f.)
   (c) 4971 (2 s.f.) **(d)** 0.099 99 (2 s.f. and 3 s.f.)
2. The number of people at a concert is stated as 21 200, correct to 3 significant figures. What is the largest and the smallest possible number of people at the concert?

5.2 Limits of accuracy

A. Upper bound and lower bound of rounded number

In Section 5.1, we have learnt how to round a given number off according to the number of decimal places or significant figures.
We have also deduced the possible actual values a given rounded number could be prior to rounding.

Upper and lower bounds of a rounded number

A number x, when rounded to 1 decimal place, is 14.5.

1. Give three possible actual values of x.
2. What is the smallest possible actual value of x before it is rounded to 1 decimal place?
3. Bernard says that the largest possible value of x before rounding is not 14.55, but 14.54. Do you agree with him? Why or why not?

From the above Class Discussion, we see that 14.45 is the smallest possible actual value of x. That is, x is *greater than or equal to* 14.45. On the other hand, we are not able to identify the largest possible value since x can be 14.549 or 14.5499 and so on. We can only say that x is *less than* 14.55. We can represent the range of possible values of x as $14.45 \leqslant x < 14.55$.

Attention

In Chapters 2 to 4, we were introduced to the symbols '>', '<', '≥' and '≤', which are used to compare two numbers. When the letter x is used in place of a number here, it represents *any number* that is larger than or equal to the lower bound and smaller than the upper bound.

The values 14.45 and 14.55 are known as the **lower bound** and **upper bound** of x. In general,

lower bound $\leqslant x <$ *upper bound*

actual value of x is *greater than or equal* to the *lower bound*

actual value of x is *less than* the *upper bound*

Exercise 5A

For each of the following rounded numbers, state the range of possible actual values.

(a) $a = 1.6$
(b) $b = 0.6$
(c) $c = 1.30$

B. Upper bound and lower bound of calculated value

When one or more rounded numbers are used in a calculation, the result also lies within a range of possible values. In this section, we will learn how to determine the upper and lower bounds of this range of values.

Determining the upper and lower bounds of calculated value

The upper and lower bounds of two numbers, p and q, are given in Table 5.1.

	Lower bound	Upper bound
p	12.5	13.5
q	0.35	0.45

Table 5.1

1. Copy and complete Table 5.2. Round each answer to 2 decimal places.

Calculation	$p = 12.5$, $q = 0.35$	$p = 12.5$, $q = 0.45$	$p = 13.5$, $q = 0.35$	$p = 13.5$, $q = 0.45$
$p + q$	$12.5 + 0.35 = 12.85$	$12.5 + 0.45 =$		
$p \times q$			$13.5 \times 0.35 = 4.73$	$13.5 \times 0.45 =$
$p - q$	$12.5 - 0.35 = 12.15$			$13.5 - 0.45 = 13.05$
$p \div q$		$12.5 \div 0.45 =$		

Table 5.2

2. In Table 5.2, circle the largest and smallest values of each calculation.
3. From the circled values in Table 5.2, complete Table 5.3 to summarise how the largest and smallest values of each calculation is obtained using the upper and lower bounds of p and of q.
[p_{lb} and p_{ub} as lower and upper bounds of p, q_{lb} and q_{ub}: lower and upper bounds of q]

	Smallest value	Largest value
Addition $p + q$	$p_{lb} + q_{lb}$	$+$
Multiplication $p \times q$	$\times\ q_{lb}$	$\times$
Subtraction $p - q$	$-$	$p_{ub} - q_{lb}$
Division $p \div q$	$\div$	$p_{ub} \div$

Table 5.3

Attention

The largest and smallest values are the upper and lower bounds of the result of each operation

Similar and Further Questions
Exercise 5A
Questions 8(a)–(d)

Worked Example 4

Determining the upper and lower bounds of calculated value

A rectangle has a length $l = 16$ cm and a breadth $b = 7$ cm, both rounded to the nearest cm. Determine the lower and upper bounds of the area of the rectangle.

Solution

$15.5 \text{ cm} \leqslant l < 16.5 \text{ cm}$

$6.5 \text{ cm} \leqslant b < 7.5 \text{ cm}$

Lower bound of area of rectangle

$= 15.5 \times 6.5$

$= 100.75 \text{ cm}^2$

Upper bound of area of rectangle
$= 16.5 \times 7.5$
$= 123.75 \text{ cm}^2$

$\therefore 100.75 \text{ cm}^2 \leqslant$ area of rectangle $< 123.75 \text{ cm}^2$

Exercise 5A
Questions 16, 17, 22

1. Ali takes 0.57 hours, rounded to the nearest 0.01 hour, to travel from Town A to Town B. If he travels at an average speed of 23.4 km/h, rounded to the nearest 0.1 km/h, determine the lower and upper bounds of the distances between Towns A and B. Leave your answers to 3 significant figures.
2. A ribbon has a length of 50 m rounded to the nearest m. Nadia wishes to cut the ribbon into shorter lengths of 22 cm rounded to the nearest cm. What is the smallest possible number of ribbons she can obtain?

1. What do I already know about rounding off whole numbers and decimals that could help me to round off numbers to a given number of significant figures?
2. How is rounding off a number to a given place value different from rounding off to a given significant figure?
3. What do I know about rounding off numbers that could help me in identifying the upper and lower bounds of the possible actual values a rounded number could have been?
4. What do I already understand about the results of basic mathematical operations that will allow me to determine the upper and lower bounds of a value calculated using rounded numbers?

Basic | Intermediate | Advanced

Exercise 5A

1. Round off 698 352 to the nearest
   (a) 10, (b) 100, (c) 1000, (d) 10 000.

2. Round off 44 974.8 to the nearest
   (a) 10, (b) 100, (c) 1000, (d) 10 000.

3. Correct 45.7395 to
   (a) 1 decimal place,
   (b) the nearest whole number,
   (c) the nearest 0.01,
   (d) the nearest thousandth.

4. Correct 7.697 146 to
   (a) the nearest whole number,
   (b) 2 decimal places,
   (c) the nearest tenth,
   (d) the nearest 0.000 01.

5. State the number of significant figures in each of the following.
   (a) 39 018
   (b) 0.028 030
   (c) 700.406 00
   (d) 2900 (to the nearest 10)
   (e) 2900 (to the nearest 100)
   (f) 2900 (to the nearest whole number)

Exercise 5A

6. Round off each of the following to the number of significant figures given in the brackets.
   (a) 728 (2 s.f.) (b) 503.98 (4 s.f.)
   (c) 0.003 0195 (4 s.f.) (d) 6396 (2 s.f. and 3 s.f.)
   (e) 9.9999 (3 s.f.) (f) 8.004 (3 s.f.)

7. For each of the following rounded numbers, state the range of possible actual values.
   (a) $a = 46$
   (b) $b = 12.6$
   (c) $c = 0.40$
   (d) $d = 2950$ (rounded to nearest 10)
   (e) $e = 300$ (rounded to nearest 10)
   (f) $f = 4500$ (3 s.f.)

8. Given that $1.25 \leqslant a < 1.35$, $15 \leqslant b < 25$ and $0.675 \leqslant c < 0.685$, determine the upper and lower bounds of
   (a) $a + b$, (b) $b - a$,
   (c) $a \div c$, (d) $c \times b$.
   Leave your calculated answers to 3 significant figures where necessary.

9. Pakistan's population was 235 820 000 in 2022. This value has been rounded to the nearest 10 000. What is the largest and the smallest possible value of Pakistan's population in 2022?

10. Round off
    (a) 4.918 m to the nearest 0.1 m,
    (b) 9.71 cm to the nearest cm,
    (c) \$10.982 to the nearest ten cents,
    (d) 6.489 kg to the nearest $\frac{1}{100}$ kg.

11. OPEN Q A swimmer training for a 100 m freestyle event said that his timing for one trial was 47.4 s. If his actual timing was recorded to 0.01 s, give three possible values of his actual timing.

12. The number 143 000 is correct to x significant figures. Write down the possible values of x.

13. The floor area of Dolmen City in Karachi, Pakistan, is stated as 320 000 m^2. This value has been rounded to 3 significant figures. What is the largest and the smallest possible integer value of the area of Dolmen City?

14. OPEN Q The diameter of a wire is measured and found to be 0.120 cm, correct to 3 significant figures. State three possible values of the actual diameter of the wire.

15. The number 21 X09 is equal to 22 000, correct to 2 significant figures. Find the value of X if 21 X09 is a perfect square.

16. A cube has sides measuring 160 mm correct to the nearest 10 mm. What are the lower and upper bounds of the volume of the cube? Express your answer in cm^3.

17. A rectangular sheet of vanguard has a length of $l = 19.0$ m and breadth $b = 1.8$ m, both rounded to the nearest 0.1 m. Joyce wants to cut the vanguard sheet into squares of sides 10.0 cm, rounded to the nearest 0.1 cm. Determine
    (i) the lower and upper bounds of the area of the vanguard sheet,
    (ii) the largest possible number of squares Joyce can cut from the sheet of vanguard sheet.

18. Cheryl says that 5192.3 is equal to 519 when rounded off to the nearest 10. She drops the '2' because it is less than 5. Do you agree? Explain your answer.

19. Yasir says that 26.97 is equal to 27 when rounded off to 1 decimal place because he thinks that 27.0 is the same as 27. Do you agree? Explain your answer.

20. David says that 0.019 95 is equal to 0.02 when rounded off to 3 significant figures. Why does David think that 0.02 has 3 significant figures? Explain whether his answer is correct or wrong and why.

Exercise 5A

21. Given the rounded values $a = 14.55$, $b - 9.6$ and $c = 21$, determine the upper and lower bounds of $\frac{180 \times b}{a \times b - c}$. Leave your answers to 3 significant figures.

22. The range of values for the perimeter of a rectangle is 72.5 cm ≤ perimeter < 74.7 cm. If the length of the rectangle is 12 cm rounded to the nearest cm, what is the width of this rectangle rounded to the nearest 0.1 cm?

5.3 Approximation and approximation errors in real-world contexts

A. Different types of rounding in real life

In mathematics, we usually round off a number using the 5 rules listed in the previous section. However, in real life, this may not be so.

Rounding in real life

1. Suppose 215 students and 5 teachers are going for an excursion by bus and each bus can only carry 30 passengers. How many buses are required? Why?
2. Suppose you design a lift that carries a maximum load of 897 kg. You are told to indicate the maximum load correct to the nearest 100 kg. What should this maximum load be? Why?

Similar and ...
Exercise 5B
Questions ...

B. Rounding errors

Rounding off **non-exact values** in *intermediate working* may result in **follow-through errors**. Let us investigate the types of follow-through errors that rounding off can lead to.

The missing 0.1% votes

Albert, Nadia and Shaha were nominated to run for President of the Mathematics Society. Table 5.4 shows the votes that they received during the election.

Candidate	Number of votes	Percentage of votes
Albert	187	62.3%
Nadia	52	17.3%
Shaha	61	20.3%
Total	**300**	**99.9%**

Table 5.4

Attention

In practice, we will just put 100% for the total percentage of votes in order not to cause confusion.

1. The total percentage of votes is only 99.9%. What has happened to the missing 0.1% of the votes? Explain your answer.
2. Given that a new member voted for Albert, calculate the percentage of votes each candidate receives from the 301 votes in a similar way (i.e. correct to 1 decimal place).
   Why is the total percentage of votes more than 100%?

In the above Investigation, the percentage of votes for each candidate is given to 1 decimal place (or 3 significant figures), so the total percentage of votes is 100% only when corrected to 2 significant figures. 99.9% = 100% (to 2 s.f.).

However, if we had written the percentage of votes for each candidate correct to 2 decimal places (or 4 significant figures), then

$$
\begin{aligned}
62.33\% + 17.33\% + 20.33\% &= 99.99\% \\
&= 100\% \text{ (to 3 s.f.)}.
\end{aligned}
$$

In other words, if we want the final answer to be accurate to 3 significant figures, the *non-exact values* in the *intermediate steps* should be given to at least 4 significant figures.

This kind of errors, which are due to rounding errors in the intermediate steps, are called *follow-through errors*.

Follow-through errors can occur if [illegible] in [illegible] are not rounded off to a sufficient degree of accuracy.

Attention

To avoid follow-through errors, intermediate values should be given to 4 or 5 significant figures for [illegible]. For *accuracy purpose*, we always use the values stored in the calculator to compute the next step of the working.

There is no need to round off **exact values**. But if an exact value has many digits, we can still round it off to 4 or 5 significant figures if it is in an intermediate step, or 3 significant figures if it is the final answer.

In the next Worked Example, we will learn how to present the intermediate working for non-exact values.

Worked Example 5

Significant figures in intermediate working

The area of a square is 131 cm². Find
(i) the length, **(ii)** the perimeter,
of the square.

(i) Length of square $= \sqrt{131}$ cm
$= 11.4$ cm (to 3 s.f.)

(ii) Perimeter of square $= 11.446$ cm $\times 4$
$= 45.8$ cm (to 3 s.f.)

Attention

Give non-exact answers to *3 significant figures* if a question does not specify the degree of accuracy.

Attention

For (ii), if we write 11.4 cm × 4, a rounding error will occur. To be more accurate, we should use the calculator value of $\sqrt{131}$ to find the perimeter.

The area of a square is 105 cm². Find
(i) the length, **(ii)** the perimeter,
of the square.

C. Difference between value with specified degree of accuracy and approximate value

1. The population of City A is *approximately 500 000*. Can the exact population size be
   (i) 450 000, (ii) 449 999?
2. The population of City B is *equal to 500 000 (to 1 s.f.)*. Can the exact population size be
   (i) 450 000, (ii) 449 999?

From the above Thinking Time, we observe that there is a difference between 'approximately 500 000' and 'equal to 500 000 (to 1 s.f.)'. It is possible for the exact population size of City A to be 449 999, but we round it up to approximately 500 000 (or half a million). But if the exact population size of City B is 449 999, then it *cannot* be equal to 500 000 (to 1 s.f.).

In Worked Example 5, can we write $\sqrt{131}$ cm ≈ 11.4 cm, where the symbol ≈ means 'is approximately equal to'?

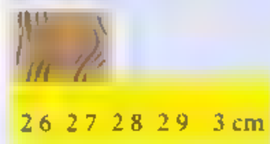
In *mathematics working*, we avoid writing $\sqrt{131}$ cm ≈ 11.4 cm because we do not know how accurate 11.4 cm is.

Instead, we always specify the degree of accuracy, such as the number of significant figures, e.g. $\sqrt{131}$ cm = 11.4 cm (to 3 s.f.).

However, we use the ≈ sign when we *estimate* an unknown quantity in real life, especially when the actual value is not known and we are unable to specify its degree of accuracy, e.g. distance ≈ 10 km (or estimated distance = 10 km).

Information

We can only measure up to half the smallest marking of a ruler. E.g. we say that the object below (see picture) is 2.75 cm long (accurate to ±0.05 cm). Therefore, if we say that the length of an object is *approximately* 2.75 cm, it does not mean it is accurate to 3 s.f. In this case, we can only measure up to 0.05 cm, i.e. we cannot even measure 2.74 or 2.76 cm.



1. What do I already know about approximation in the real world that could guide my learning in this section?
2. What have I learnt in this section that I am still unclear of?

5.4 Estimation and estimation errors in real-world contexts

In the previous sections, we have learnt how to **approximate** a *given* number by rounding it off to get an approximate value.

In this section, we will learn how to **estimate** an *unknown* quantity in real life. We usually give an estimated value by saying that the quantity is approximately a certain value.

We will also discuss the importance and usefulness of estimation.

A. Estimation of computations

Sometimes, we may key in the wrong value when using a calculator and obtain a wrong answer.

We can use estimation to check if an answer obtained from a calculator is *reasonable* or obviously wrong.

Worked Example 6

Using estimation to check reasonableness of answer

Joyce used a calculator to evaluate 31.5 + 9.87 − 2.1 and obtained the answer 392.7. Without doing the actual calculation, use estimation to check whether Joyce's answer is reasonable.

Then use a calculator to evaluate 31.5 + 9.87 − 2.1. Is your estimated value close to the actual value?

Solution

$$31.5 + 9.87 - 2.1 \approx 32 + 10 - 2$$
$$= 40$$

∴ Joyce's answer, 392.7, is not reasonable.

Using a calculator, the actual answer is 39.27.

Hence, the estimated value, 40, is close to the actual value of 39.27.

Attention

We use the approximation sign ≈ when it is not important to specify the degree of accuracy. e.g. 31.5 ≈ 32.

Exercise 5B

1. Ali used a calculator to evaluate 798 × 195 and obtained the answer 15 561.
   Without doing the actual calculation, use estimation to determine whether Ali's answer is reasonable.
   Then use a calculator to evaluate 798 × 195. Is your estimated value close to the actual value?

Information

There is another method for Question 1. look at the last digits of the two numbers and compare their product with the last digit of the answer obtained.

2. Estimate each of the following without using a calculator. Then use a calculator to find the actual values. Are your estimated values close to the actual values?

   (a) 5712 ÷ 297 (b) $\sqrt{63}\times\sqrt[3]{129}$

3. The driving distance between Islamabad and Kohat is about 250 km. Estimate the duration, in hours, taken to drive from Islamabad to Kohat at an average speed of 80 km/h.

B. Shopping with a different currency

Suppose you are shopping overseas and wish to convert the prices quoted in the local currency to Pakistani rupees. Estimation provides an easy way to do so.

Worked Example 7

A wallet costs 225 Thai baht (THB). The conversion rate is THB 1 = PKR 8.415 294.
Without using a calculator, estimate the price of the wallet in PKR.

Solution

THB 1 ≈ PKR 8.50
THB 200 ≈ PKR 1700
THB 50 ≈ PKR 425
THB 25 ≈ PKR 200
THB 225 = THB 200 + THB 25
≈ PKR 1700 + PKR 200 = PKR 1900
∴ price of wallet ≈ PKR 1900

> **Problem-solving Tip**
> THB 1 = PKR 8.415 294 is not easy to remember. It is also difficult to use this number to estimate the price of the wallet. Thus we use an approximate value for THB 100, i.e. THB 100 = PKR 8 50.

A pair of earrings costs 25 000 Indonesian rupiah (IDR).
The conversion rate is IDR 1000 = PKR 19.342.
Without using a calculator, estimate the price of the pair of earrings in PKR.

C. Value for money

8

Value for money

You are in a supermarket with your neighbour and she sees the following options for the same brand of coffee powder.

Option A	Option B
	
200 g Coffee powder	200 g + 50 g Coffee powder
$5.80	$7.45

> **Attention**
> Although Option A seems cheaper (since \$5.80 < \$7.45), Option A contains less coffee powder than Option B.
> To find out which option gives the **better value for money**, compare the cost of coffee powder for an *equal amount* of coffee powder for both options.

Without using a calculator, how do you help her decide which option gives the *better value for money*?

Solution

Method 1:

Option A:
200 g costs about $6.
100 g costs about $3.
50 g costs about $1.50.
250 g = 200 g + 50 g costs about $6 + $1.50 = $7.50.

Option B:
250 g costs \$7.45.

Does this mean that Option B gives the better value for money since $\$7.45 < \$7.50$?
In this case, we have **overestimated** the cost of 250 g of coffee powder in Option A (see **Method 2**).

Method 2:
Option A:
200 g costs *\$5.80* ≈ \$6.
100 g costs about \$3.
50 g costs about \$1.50.
250 g = 200 g + 50 g costs about *\$5.80* + \$1.50 = \$7.30.

Option B:
250 g costs \$7.45.

∴ Option A gives the better value for money since $\$7.30 < \$7.45$.

Problem-solving tip

This Worked Example shows that it is possible to [illegible] or [illegible] the value of a quantity, which may then affect our decision.
If two estimated values are [illegible] to each other, we may have to use another method of estimation (in this case, [illegible]) or use the actual values to compute.

Similar and [illegible] Questions
Exercise 5B Questions 14, [illegible]

Without using a calculator, decide which option gives the better value for money.

Option A	Option B
	
300 ml Olive oil \$8.80	300 ml + 50 ml Olive oil \$10.40

D. Estimation of larger quantity using smaller quantity

The Greek mathematician, Archimedes, estimated that 8×10^{63} grains of sand were required to fill the universe.

Archimedes did not make a wild guess or count every grain. Instead, he made use of an important **estimation strategy**: using a smaller quantity to estimate a larger quantity.

First, he counted the number of grains of sand required to fill a spoon.

Next, he estimated the number of spoons of sand required to fill a room, the number of rooms of sand required to fill a stadium, and so on.

Using smaller quantity to estimate larger quantity

How can you estimate the amount of money in a tank that is completely filled with PKR coins?

Even if you were able to obtain an identical tank, it would be too troublesome to get so many PKR 1 coins. A useful estimation strategy is to fill a smaller box with PKR 1 coins then count the number of coins You may wish to repeat this twice and take the average of the three trials.

Use the above strategy to estimate the number of PKR 1 coins in a tank with dimensions of 50 cm by 23 cm by 13 cm.

Estimation of area

Estimate the ratio of the area of the shaded region to that of the unshaded region in the figure below.

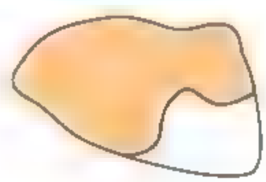
$\therefore$ ratio of area of shaded region to that of unshaded region

$\approx 3 : 1$

Problem-solving Tip

To estimate the area, we divide the shaded region (using dotted lines) into areas that are approximately equal to the area of the unshaded region.
Since the unshaded area is on the right side of the figure, start dividing the shaded region from the right side to obtain a more accurate estimate.

Estimate the percentage of the figure on the right that is shaded.



1. What do I already know about estimation in the real world that could guide my learning in this section?
2. How is estimation different from approximation?
3. What have I learnt in this section or chapter that I am still unclear of?

Exercise 5B

If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures.

1. What is the greatest number of sweets that can be bought with \$2 if each sweet costs 30 cents?

2. Suppose you are an engineer tasked to design a multi-storey car park. The height restriction for vehicles entering the car park is calculated to be 2.51 m. A sign indicating the maximum height, correct to the nearest metre, is to be placed at the entrance. What should the maximum height be shown as? Explain your answer.

3. The items you buy at a supermarket amount to \$17.69. After informing the cashier that you will be paying in cash, she rounds *down* the amount to the nearest 5 cents instead of rounding the amount to the nearest 5 cents.
   (i) What is the difference between 'rounding *down* the amount to the nearest 5 cents' and 'rounding the amount to the nearest 5 cents'?
   (ii) Why does the cashier round *down* the amount to the nearest 5 cents instead of rounding the amount to the nearest 5 cents?

4. The area of a square is 264 cm². Find
   (i) the length, (ii) the perimeter,
   of the square.

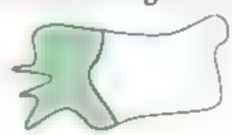
5. The area of a rectangle is 25.6 m². If the length of the rectangle is 12 m, find
   (i) the breadth,
   (ii) the perimeter,
   of the rectangle.

6. Shaha used a calculator to evaluate $218 \div 31$ and obtained the answer 70.3.
   Without doing the actual calculation, use estimation to determine whether Shaha's answer is reasonable. Then use a calculator to evaluate $218 \div 31$. Is your estimated value close to the actual value?

7. Estimate each of the following without using a calculator. Then use a calculator to find the actual values. Are your estimated values close to the actual values?
   (a) 2013×39 (b) $\sqrt{145.6} \div \sqrt[3]{65.4}$

8. (i) Express 3.612 and 29.87 correct to 2 significant figures.
   (ii) Use your answers in part (i) to estimate the value of $3.612 \div 29.87$.
   (iii) Use a calculator to evaluate the value of $3.612 \div 29.87$.
   (iv) Is your estimated value close to the actual value?

9. Estimate the ratio of the area of the shaded region to that of the unshaded region in the figure.



10. (i) Find the area of a rectangular field measuring 27.04 m by 20.21 m.
    (ii) How much would it cost to spray insecticide on the field if the rate is \$0.70 per square metre? Give your answer to the nearest dollar.

11. A car travels 274 km. It travels an average of 9.1 km on a litre of petrol. Write down an expression that you can use to mentally estimate the number of litres used.

Exercise 5B

12. A shopkeeper makes the following order from a wholesaler:

Item	Quantity	Cost per item ($)
Skirts	32	19
Belts	18	9
White shirts	49	22
Black blouses	61	18
Grey leggings	52	11

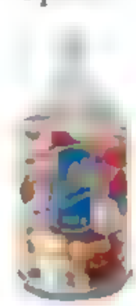
Show how you estimate the total amount of money that the shopkeeper has to pay, giving your answer correct to the nearest hundred dollars.

13. A bag costs MYR 25. The conversion rate is MYR 1 = PKR 64.328 70. Without using a calculator, estimate the price of the bag in PKR.

14. Without using a calculator, decide which option gives the better value for money.

Option A

Option B



15. Which of the following is likely to be the mass of an ordinary car?
(a) 15 kg
(b) 150 kg
(c) 1500 kg
(d) 15 000 kg

16. The picture shows a man of height 1.7 m standing in front of a block of flats. Estimate the height of the block of flats, leaving your answer to the nearest metre. Show your working clearly.

[illegible] A handbag costs 26 700 Korean won (KRW). The conversion rate is PKR 1 = KRW 4.516 097. Without using a calculator, estimate the price of the handbag in PKR.

[illegible] Shop A sells a pair of shoes at $80.50 before a 20% discount while Shop B sells the same pair of shoes at $69.50 before a 10% discount. Without using a calculator, use estimation to decide which shop to buy the shoes from. Show your working clearly.

Although we use mathematics as a way to measure or quantify objects more accurately, it is sometimes faster and more practical to estimate the quantity, or to express the quantity using an approximate value. For example, the has shown that in the real world, there are times when an approximate value serves as a satisfactory model for the situation.

Enrico Fermi is a well-known physicist who was involved in the Manhattan Project, which was established in World War II to produce the first nuclear weapons. During the Trinity test, he estimated the power of an atomic bomb explosion by finding the distance travelled by strips of paper that he dropped from his hand during the blast His guess of 10 kilotons of TNT was not too far from the now-accepted value of 18.6 kilotons of TNT obtained by sophisticated methods and instruments. His low-cost method gave an adequate estimate of the power of the atomic bomb! This is an excellent example of how an estimated measure can model the real world reasonably well. Fermi's power of estimation is legendary, and today, there is a class of very interesting estimation problems available. Search the Internet for some Fermi problems and solve them using what you have learnt.

Can you think of other situations in which an estimation or approximation is both a faster and easier way to either obtain a measure of a quantity or model a real-world situation?

1. **Five rules to identify digits which are significant**
   Rule 1: All non-zero digits are significant.
   Rule 2: All zeros between non-zero digits are significant.
   Rule 3: In a decimal, all zeros before a non-zero digit are not significant.
   Rule 4: In a decimal, all zeros after a non-zero digit are significant.
   Rule 5: The zeros at the end of a whole number may or may not be significant, depending on how the number is rounded off.
   - Give an example of each of the rules above.
   - Give an example of a rounding off that does not follow the rules above.

2. (a) The **lower and upper bounds** of a rounded number are the limits of the possible actual values the rounded number could have been, i.e. if x has been rounded, then the possible actual values of x can be described using

$$\text{lower bound} \leqslant x < \text{upper bound}.$$

   - Give an example of a rounded number and write down the lower and upper bounds of this number.

(b) If one or more rounded numbers are used in a calculation, the result of the calculation also lies within a range of possible values. The lower and upper bounds of this range of values are determined by the type(s) of mathematical operation(s) involved in the calculation:

	Lower bound	Upper bound
Addition $p + q$	$p_{lb} + q_{lb}$	$p_{ub} + q_{ub}$
Multiplication $p \times q$	$p_{lb} \times q_{lb}$	$p_{ub} \times q_{ub}$
Subtraction $p - q$	$p_{lb} - q_{ub}$	$p_{ub} - q_{lb}$
Division $p \div q$	$p_{lb} \div q_{ub}$	$p_{ub} \div q_{lb}$

- Give an example of pair of rounded numbers and determine the lower and upper bounds of the result of each of the mathematical operations above.

3. **Approximation** is the process of rounding off a given number to give an approximate value
   - Give two examples of an approximation in a real-world context.

4. **Follow-through errors** can occur if *non-exact values* in *intermediate working* are not rounded off to a sufficient degree of accuracy.
   (a) For *presentation purpose*, we usually write 4 or 5 significant figures for intermediate non-exact values if we want the final answer to be accurate to 3 significant figures.
   (b) For *accuracy purpose*, we always use the values stored in the calculator to compute the next step of the working.

5. **Estimation** is the process of guessing the value of an *unknown* quantity in real life.
   - Give two examples of an estimation in a real-world context.

CHAPTER 6

Basic Algebra and Algebraic Manipulation

Algebra is an important foundation of advanced mathematics. Mathematicians use algebra to express ideas about numbers and the relationship between numbers.

A monument honouring Muhammed bin Musa Al-Khwarizmi (780 – 850), known as the father of algebra, is found in Khiva, Uzbekistan. The word algebra comes from an Arabic word (*al-jabr*, which literally means restoration). 'The Compendious Book on Calculation by Completion and Balancing', a mathematical text published by Al-Khwarizmi, established algebra as a mathematical discipline independent of geometry and arithmetic. It is considered the foundational text of modern algebra.

In this chapter, we will explore the basic ideas in algebra and learn to express mathematical ideas using algebra in order to model real-world situations around us.

Learning Outcomes

What will we learn in this chapter?

- What algebraic and linear expressions are
- How to add, subtract, expand, factorise and simplify linear expressions
- Why algebra has useful applications in real-world contexts

Introductory Problem

Let me introduce myself. I am Cheryl, a psychic who can read your mind.
Let us play a game.

Step 1: Think of a two-digit positive integer that is greater than 20.
Step 2: Add the two digits together.
Subtract the sum of the two digits from your original number to get a new number.
Add the two digits of this new number together to obtain the final number.
Step 5: Focus on this final number while I read your mind.

This will take a few seconds…

The final number in your mind is 9. Am I correct?

(i) Did all your classmates also obtain 9 as their final number?
(ii) Discuss with your classmates how Cheryl knew all your final numbers.

You may have tried different examples in the above activity. Can you explain or prove why the final answer would always be 9?
In this chapter, we will learn to solve such problems using algebra.

6.1 Basic algebraic concepts and notations

A. Using letters to represent specific unknown numbers

In primary school, we have used symbols to represent a missing number. For example,

$$● + 5 = 7.$$

● represents an unknown and we can solve for it.

We can also use a letter to represent the unknown. For example,

$$n + 5 = 7.$$

Letters, such as n, are used in algebra to represent different kinds of numbers.
Let us look at an example. There are some sweets in a box and Bernard ate three of them.

Table 6.1 shows some possible numbers of sweets in the box initially, and the corresponding number of sweets left.

Information

In standard English, x is called a letter, *not* an alphabet. The English alphabet contains 26 letters from a to z.

Initial number of sweets	Remaining number of sweets
100	97
53	50
41	38

Table 6.1

We see that the remaining number of sweets is 3 less than the initial number in the box, for any possible pair of numbers.

If we do not know the initial number of sweets, we can represent it with x. When 3 sweets are eaten, there will be $(x - 3)$ sweets left.

Here, we use the letter x to represent a **specific unknown number**. $(x - 3)$ is an example of an **algebraic expression**.

If the number of sweets left is y, what would the algebraic expression that represents the initial number of sweets in the box be?

Without further information, we will not be able to find the unknown x. However, if we know that there are 7 sweets left, then we can find the unknown x by drawing a model and forming an equation as shown in Fig. 6.1.

Fig. 6.1

$$\begin{aligned} x - 3 &= 7 \\ x &= 7 + 3 \qquad \text{solve by looking at model} \\ &= 10 \end{aligned}$$

Information

We can view algebra as the study of procedures to solve certain kinds of problems. (Not all problems can be solved by algebra.)

We use letters as specific unknown numbers in algebra when we want to solve problems like the above. We will learn how to solve algebraic equations in the next chapter.

B. Using letters to represent generalised numbers

However, not all letters in algebra represent specific unknown numbers.
We also use letters as **generalised numbers** in algebra to express a general law or property in arithmetic.
For example, when we add **two numbers in arithmetic**, the order of the two numbers does not matter, e.g. $4 + 3 = 3 + 4$, or $(-2) + 5 = 5 + (-2)$.
To generalise this property of addition, we use a and b to represent any two numbers and we write:

$$a + b = b + a.$$

Attention

The property $a + b = b + a$ is called the *commutative law of addition*.

In this case, a and b are not specific unknown numbers but generalised numbers used to represent any numbers. We cannot find the value of a and of b

Information

We can also view algebra as generalised arithmetic.

Letters as generalised numbers

1. When we multiply two numbers in arithmetic, e.g. 2×3, does the order of the two numbers matter? Explain your reasoning.
   - (i) Write down this property for 2×3.
   - (ii) Write down this property using letters.
2. When we subtract two numbers in arithmetic, e.g. $5 - 2$, does the order of the two numbers matter? Explain your reasoning.
   - (i) Write down this property for $5 - 2$ using the notation for 'not equal to': $\neq$.
   - (ii) Write down this property using letters.
3. Do the same for the division of two numbers.

C. Using letters to represent variables

In primary school, we have learnt that the perimeter of a square is 4 times its length. This relationship between the two quantities, perimeter P and length l, of a square can be represented using letters:

$$P = 4 \times l \quad \text{or} \quad P = 4l.$$

In this case, P and l are called **variables** because as l *varies* (i.e. changes in value), P also varies (unlike specific unknown numbers which have specific values). Moreover, if a value of l is given, we can find the value of P, and vice versa. In other words, we use letters as variables in algebra when we want to express a relationship between two or more quantities.

Information

We can also view algebra as the study of relationships between quantities.

Using letters, express the relationship between
- (a) the area A of a square and its length l,
- (b) the area A of a rectangle and its length l and breadth b,
- (c) the perimeter P of a rectangle and its length l and breadth b.

We will learn more about relationships among quantities in Chapter 7.

D. Algebraic notations

We have written expressions such as $4 \times l$ and $l \times l$ earlier on. Algebraic notations allow us to write them in a shorter way. Table 6.2 shows a summary of some algebraic notations and their meanings.

Notation	Meaning
$3x$	$3 \times x$ or $x + x + x$
xy	$x \times y$
$\frac{x}{y}$	$x \div y$ or $x \times \frac{1}{y}$
x^2	$x \times x$
x^3	$x \times x \times x$

Table 6.2

There are certain conventions to write algebraic notations in a concise and precise manner to communicate mathematical ideas clearly. For example,
- we write $3 \times x$ as $3x$, *not* $x3$,
- we usually write $1 \times x$ as x, *not* $1x$;
- we write the letter x differently to distinguish between the letter x and the multiplication sign $\times$ (see Table 6.2)

Interpreting meanings of algebraic notations

Discuss and write down the meaning of each of the following notations.

(a) $2n$ (b) ab (c) $\frac{a}{b}$

(d) n^2 (e) $4x^3$ (f) x^2y

Let us consider the notation $2n$ in part **(a)** of the above Class Discussion.
We know that $2n = 2 \times n = n + n$. But is $2n$ different from $2 + n$ or n^2?
Let us investigate with the help of a spreadsheet.

Comparing algebraic notations

1. Create a spreadsheet as shown. Key in the formulae in the cells B2, C2, D2, E2 and F2 (* means × in the spreadsheet).

	A	B	C	D	E	F	G	H
1	n	$2n$	$2 + n$	$n + n$	$n \times n$	n^2		
2	1	=2*A2	=2+A2	=A2+A2	=A2*A2	=A2^2		
3	2							
4	3							
5	4							
6	5							

Table 6.3

You should obtain the following results after you have entered the above formulae in the cells. (Do not type the values in B2 to F2 directly!)

	A	B	C	D	E	F	G	H
1	n	$2n$	$2+n$	$n+n$	$n\times n$	n^2		
2	1	2	3	2	1	1		
3	2							
4	3							
5	4							
6	5							

Table 6.4

2. The cell B2 shows the number 2 because when $n = 1$ (in the cell A2), according to the formula 2*A2, $2n = 2 \times n = 2 \times 1 = 2$.
Explain how the spreadsheet obtained the numbers in the cells C2 to F2.

3. Select the cells B2 to F2. To copy the respective formulae in B2 to F2 down the rows, move the cursor to the bottom right corner of the cell F2 until a '+' appears. Then click on the '+' and drag downwards to F6. You should obtain the following results for rows 3 and 4.

	A	B	C	D	E	F	G	H
1	n	$2n$	$2+n$	$n+n$	$n\times n$	n^2		
2	1	2	3	2	1	1		
3	2	4	4	4	4	4		
4	3	6	5	6	9	9		
5	4							
6	5							

Table 6.5

Copy and complete Table 6.5 for Rows 5 and 6.

4. Compare the values in Columns B to F and answer the following questions. For each question, explain your answer.
   (i) Is $2n = 2 + n$? (ii) Is $2 + n = n + n$? (iii) Is $2n = n + n$?
   (iv) Is $2n = n \times n$? (v) Is $n^2 = n \times n$? (vi) Is $n^2 = 2n$?

5. In cell G1, type $2n^2$; in cell G2, key in the formula 2*A2^2.
In cell H1, type $(2n)^2$; in cell H2, key in the formula (2*A2)^2.
Copy the formulae in G2 and H2 all the way down until G6 to H6 (see Step 3).
Copy and complete Table 6.5 for Columns G and H.

6. (i) Compare the values in Columns G and H. Is $2n^2 = (2n)^2$? Explain your answer.
   (ii) $2n^2$ means $2 \times n \times n$. What does $(2n)^2$ mean?

From the Investigation on pages 133 and 134, we obtain the following results which are *true in general* for any n.

Meanings of some algebraic notations

- $2n \neq 2 + n$ but $2n = 2 \times n = n + n$
- $n^2 \neq 2n$ but $n^2 = n \times n$

Attention

$n = 2$ is an exception to these results.
But in general, $2n \neq 2 + n$ and $n^2 \neq 2n$ for all other values of n.

E. Algebraic expressions

In Section 6.1A, we came across the **algebraic expressions** $(x - 3)$ and $(y + 3)$.

In general, an algebraic expression is an expression containing letters, numbers and/or operations.
An algebraic expression has *no equal sign*.

Let us consider the algebraic expression $2x - y + 7$.

There are 3 **terms** in this algebraic expression.

$$2x - y + 7$$

x and y are **variables**. This is a *constant term*. It does not contain any variables.

Attention

In Section 6.1A, we also came across the algebraic equation $n + 5 = 7$.
An algebraic equation contains an algebraic expression on either side, or both sides, of the equal sign.

In the first term $2x$, the number 2 is called the **coefficient** of x. What is the coefficient of $-y$ in the second term?
Can an algebraic expression consist of *only one term*, e.g. $5x$?

Expressing mathematical operations and simple real-world situations using algebraic expressions

1. Copy and complete Table 6.6.

In words	Algebraic expression
(a) Sum of $2x$ and $3z$	
(b) Product of x and $7y$	
(c) Divide $3ab$ by $2c$	
(d) Subtract $6q$ from $10z$	
(e)	$(p + q) - xy$
(f)	$\frac{3+y}{5}$
(g)	$\sqrt{b} - 2c$
(h) There are three times as many girls as boys in a school. Find an expression, in terms of x, for the total number of students in the school, where x represents the number of boys in the school.	It is given that x represents the number of boys. $\therefore$ ____ represents the number of girls. Total number of students = ____
(i) Nadia's father is three times as old as her. Nadia's brother is 5 years older than her. Find an expression, in terms of y, for the sum of their ages, where y represents Nadia's age.	It is given that y represents Nadia's age. $\therefore$ Nadia's father is ____ years old. Nadia's brother is ____ years old. Sum of their ages = ____ years
(j) The ____ is ____ times as long as the ____ of the rectangle. Find an expression, in terms of b, for the perimeter and the area of the rectangle, where b represents the breadth of the rectangle.	It is given that b represents the breadth of the rectangle in m. $3b$ represents the length of the rectangle in m. Perimeter of the rectangle = ____ m Area of the rectangle = ____ m^2

Table 6.6

2. Think of three different algebraic expressions and get your classmate to interpret the mathematical relationships in each of them.

F. Evaluating algebraic expressions

Let us learn how to evaluate an algebraic expression.

Evaluating algebraic expressions

Evaluating an algebraic expression is the process of finding the value of the expression when its variables are given certain values.

Evaluating algebraic expressions

Given that $x = 5$ and $y = -3$, evaluate each of the following expressions.

(a) $3x - 2y$ **(b)** $\frac{2y}{3x}$

*Solution

(a) $3x - 2y = 3(5) - 2(-3)$
$= 15 - (-6)$
$= 15 + 6$ recall: $5 - (-2) = 5 + 2$
$= 21$

(b) $\frac{2y}{3x} = \frac{2(-3)}{3(5)}$
$= -\frac{6}{15}$
$= -\frac{2}{5}$ reduce to the simplest form

Attention

(a) $3(5)$ means 3×5, and $2(-3)$ means $2 \times (-3)$. Alternatively, you can also present your working as $3x - 2y = 3 \times 5 - 2 \times (-3)$

1. Given that $x = -2$ and $y = 4$, evaluate each of the following expressions.

(a) $5y - 4x$ **(b)** $\frac{1}{x} - y + 3$

2. Find the value of $p^2 + 3q^2$ when $p = -\frac{1}{2}$ and $q = -2$.

Is $5 + n$ or $5n$ greater in value? Explain your answer.

G. Linear expressions

A **linear expression** in *one variable* x is an algebraic expression that contains only one term in x, with or without a constant term.

Examples of linear expressions	Examples of non-linear expressions
• $2x + 7$ is a linear expression with one term in x and one constant term	• 7 is a constant
• $-\frac{1}{3}x$ is a linear expression with one term in x and no constant term (i.e. constant term is 0)	• $3x^2 - 6x$ contains a term in x^2
• $2x + 3x - \frac{9}{4}$ is a linear expression because we can add the two terms in x to get one term in x	• $5x - 7y + 2xy$ contains a term in xy
• $4x + y - 8$ is a linear expression in *two variables* x and y	

Linear expressions

Are the following linear or non-linear expressions? Explain your reasoning.

(a) $x - \frac{1}{2}$ (b) $-3y$ (c) 10 (d) $5y - y + 9$

(e) $4y + y^2$ (f) $-2y + 3x + 7$ (g) $6x - 2y + 3z + 7$ (h) $x - 4y + 8xy$

1. In what way(s) is using letters to form an algebraic expression similar to/different from using a symbol such as ●?
2. What new mathematical notations and vocabulary have I learnt in this section?

6.2 Addition and subtraction of linear terms

In this section, we will learn how to add and subtract linear terms.

A. Like terms and unlike terms

Like terms contain the same variable(s) raised to the same power. Terms that are not like terms are called **unlike terms**.

Let us consider the algebraic expression $2x - 3xy + 7 + x^2 + 5x - 4 + 2yx$.

It contains 2 variables (x and y) and 7 terms ($2x$, $-3xy$, 7, x^2, $5x$, -4 and $2yx$).

Like terms	Unlike terms
• $2x$ and $5x$ contain the same variable x of the same power • 7 and -4 are both constant terms as they do not contain any variable • $-3xy$ and $2yx$ contain the same variables because $xy = yx$	• $2x$ and $-3xy$ contain different variables (x and xy) • $2x$ and x^2 contain the same variable x, but the variables x are of different powers

Like and unlike terms

Are the following pairs of terms like or unlike terms? Explain your reasoning.

(a) $4y$ and $-y$ (b) -3 and 8 (c) $5xy$ and $5x$

(d) $7xy$ and $-2yx$ (e) $3y$ and y^2 (f) $-6x^2$ and $4x^3$

B. Addition and subtraction involving like terms with positive coefficients

In Chapter 4, we used number discs representing 1 (i.e. +1) and −1 as shown in Fig. 6.2(**a**). In this chapter, we will introduce discs representing x (i.e. $+x$) and $-x$ as shown in Fig. 6.2(**b**).

Together, we call them 'algebra discs'.

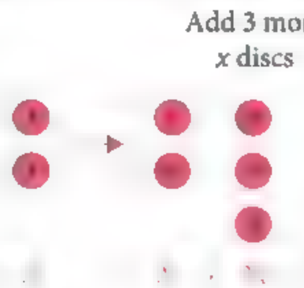
(a) **(b)**

Fig. 6.2

In Question 1(**h**) of the Class Discussion on page 136, the algebraic expression $x + 3x$ can be simplified to $4x$. *Like terms* such as x and $3x$ can be added and/or subtracted. We will now use algebra discs to illustrate this.

To illustrate $2x + 3x = 5x$, we put two discs as shown in Fig. 6.3(**a**) to represent $2x$.

To add the term $3x$, we add three more discs as shown in Fig. 6.3(**b**).



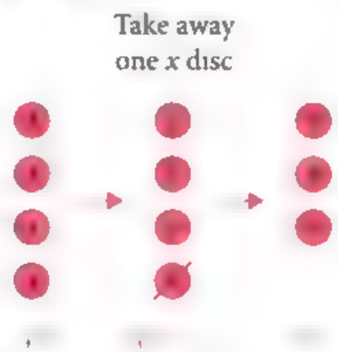
(a) **(b)**

Fig. 6.3

$\therefore\ 2x + 3x = 5x$

To illustrate $4x - x = 3x$, we put four discs as shown in Fig. 6.4(**a**) to represent $4x$.

To subtract x, we take away one disc as shown in Fig. 6.4(**b**).



(a) **(b)** **(c)**

Fig. 6.4

$\therefore\ 4x - x = 3x$

What do we get when we add or subtract unlike terms? For example, what is $2x + 3y + 1$ equal to?

As you can see in Fig. 6.5, $2x + 3y + 1$ is still $2x + 3y + 1$, i.e. the expression *cannot be further simplified* because $2x$, $3y$ and 1 are unlike terms.

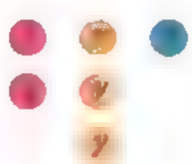


Fig. 6.5

C. Addition and subtraction involving like terms with negative coefficients

How do we perform the following operations: $(-2x) + (-3x)$, $5x + (-2x)$, $2x - 5x$ and $5x - (-2x)$?

Let us investigate using algebra discs.

What is $(-2x) + (-3x)$ equal to?

To represent the term $-2x$, we put two $-x$ discs as shown in Fig 6.6(**a**).

To add the term $-3x$, we add three more $-x$ discs as shown in Fig. 6.6(**b**).

> **Attention**
>
> $(-2x) + (-3x)$ can be written as $-2x + (-3x)$. The first pair of brackets is not necessary, but the second pair is necessary. Why?

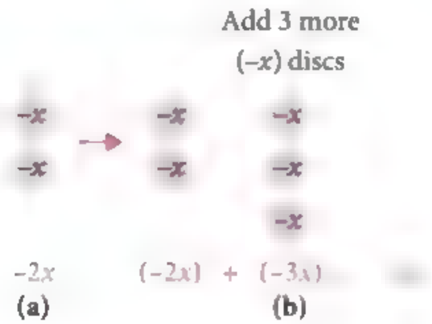


Fig. 6.6

$\therefore (-2x) + (-3x) = -5x$

In other words, for $(-2x) + (-3x)$, we add the coefficients of the two terms (i.e. -2 and -3) to get -5, which will be the coefficient of x in the sum $-5x$.

Addition and subtraction of like terms

1. Work with algebra discs to find the result of $5x + (-2x)$.
   Recall: $5 + (-2) = 5 - 2 = 3$.
   What do you observe about the results for $5x + (-2x)$ and $5 + (-2)$?

2. Work with algebra discs to find the result of $2x - 5x$.
   Recall: $2 - 5 = -3$.
   What do you observe about the results for $2x - 5x$ and $2 - 5$?

3. Work with algebra discs to find the result of $-5x - 2x$.
   Recall: $-5 - 2 = -7$.
   What do you observe about the results for $-5x - 2x$ and $-5 - 2$?

4. Work with algebra discs to find the result of $5x - (-2x)$.
   Recall: $5 - (-2) = 5 + 2 = 7$.
   What do you observe about the results for $5x - (-2x)$ and $5 - (-2)$?

From the Investigation on page 140, we observe this:

Addition and subtraction of like terms

To add or subtract like terms, we add or subtract their coefficients, which will be the coefficient of the answer. The variable remains the same.

The observations are the same as those for the addition and subtraction of real numbers.

E.g. $(-2) + (-3) = -5$;	likewise, $(-2x) + (-3x) = -5x$.
E.g. $5 + (-2) = 5 - 2 = 3$;	likewise, $5x + (-2x) = 5x - 2x = 3x$;
and $-2 + 5 = 5 - 2 = 3$;	and $-2x + 5x = 5x - 2x = 3x$.
E.g. $2 - 5 = -3$;	likewise, $2x - 5x = -3x$
E.g. $-5 - 2 = -7$;	likewise, $-5x - 2x = -7x$.
E.g. $5 - (-2) = 5 + 2 = 7$;	likewise, $5x - (-2x) = 5x + 2x = 7x$.

Worked Example 2

Adding and subtracting algebraic terms

Without using a calculator, simplify the following.

(a) $4x + (-7x)$ (b) $-9x + 8x$ (c) $-3y - 4y$ (d) $-6y - (-2y)$

Solution

(a) $4x + (-7x) = 4x - 7x$
$= -3x$

(b) $-9x + 8x = 8x - 9x$ recall: $-2x + 5x = 5x - 2x$
$= -x$ recall: $2x - 5x = -3x$

(c) $-3y - 4y = -7y$

(d) $-6y - (-2y) = -6y + 2y$
$= 2y - 6y$
$= -4y$

Exercise 6A

Without using a calculator, simplify the following.

(a) $3x + (-7x)$ (b) $19x + (-33x)$ (c) $-6x + 2x$ (d) $-35x + 12x$

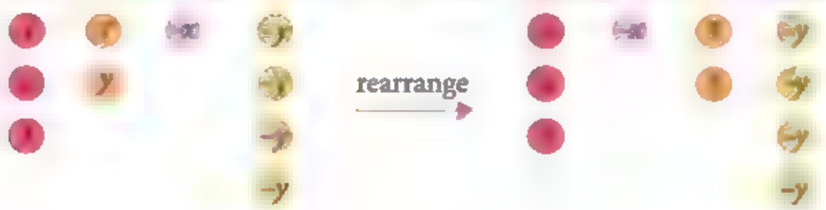
(e) $-9y - 11y$ (f) $-39y - 12y$ (g) $-8z - (-3z)$ (h) $-17z - (-35z)$

D. Addition and subtraction of algebraic terms involving two variables

How do we add or subtract linear terms involving two variables, e.g. $3x + 2y - x + (-4y)$?
We can simplify this to:

$3x + 2y - x + (-4y) = 3x + 2y + (-x) + (-4y)$ recall: $5x - 2x = 5x + (-2x)$

Now we can represent the above expression using algebra discs as shown in Fig. 6.7(**a**).
To simplify a linear expression involving two variables, we *group the like terms* together first, as shown in Fig. 6.7(**b**).



$3x + 2y + (-x) + (-4y)$
(**a**)

$3x + (-x) + 2y + (-4y) = 3x - x + 2y + (-4y)$
(**b**)

Fig. 6.7

In other words, $3x + 2y - x + (-4y) = 3x - x + 2y + (-4y)$.
rearrange

Therefore, we can shift $-x$ in the third term to the second term.

Attention

We cannot change the order of the terms without considering the minus sign. So
$3x + 2y - x + (-4y)$
$= 3x - x + 2y + (-4y)$

Worked Example

Adding and subtracting algebraic terms in 2 variables

Without using a calculator, simplify the expression $-5x + (-6y) - 3x - (-4y)$.

$-5x + (-6y) - 3x - (-4y)$
$= -5x - 3x + (-6y) - (-4y)$ group like terms
$= -8x + (-6y) + 4y$ recall: $-5x - 2x = -7x$ & $5x - (-2x) = 5x + 2x$
$= -8x + 4y - 6y$ recall: $-2x + 5x = 5x - 2x$
$= -8x - 2y$ recall: $2x - 5x = -3x$

Without using a calculator, simplify the following.

(**a**) $-4x + (-7y) - 2x - (-3y)$
(**b**) $-8y + (-3x) - (-2y) + (-4x)$
(**c**) $9x + (-2z) + 7 - (-5z) + (-x) - 4$
(**d**) $6a - (-8) - 3b + a + (-7b) + 5$

1. How do I identify like terms and unlike terms?
2. What do I already know about the addition and subtraction of real numbers that could help me learn the addition and subtraction of algebraic terms?

Exercise 6A

1. Write down an algebraic expression for each of the following statements.
   (a) Add $5y$ to the product of a and b.
   (b) Subtract 3 from the cube of f.
   (c) Multiply k by $6q$.
   (d) Divide $2w$ by $3xy$.
   (e) Subtract 4 times the positive square root of z from thrice of x.
   (f) Twice the variable p divided by the product of 5 and q.

2. Given that $x = 6$ and $y = -4$, evaluate each of the following expressions.
   (a) $4x - 7y$
   (b) $\frac{5x}{3y} + x$
   (c) $2x^2 - y^3$
   (d) $3x + \frac{x}{y} - y^2$

3. Given that $a = 3$, $b = -5$ and $c = 6$, evaluate each of the following expressions.
   (a) $a(3c - b)$
   (b) $ab^2 - ac$
   (c) $\frac{b}{a} - \frac{c}{b}$
   (d) $\frac{b+c}{a} + \frac{a+c}{b}$

4. Simplify the following.
   (a) $5x + (-8x)$
   (b) $-12x + 7x$
   (c) $13x + (-19x)$
   (d) $-28x + 6x$
   (e) $-4y - 9y$
   (f) $-16y - 3y$
   (g) $-7z - (-2z)$
   (h) $-11z - (-24z)$

5. Simplify each of the following expressions.
   (a) $5x + 22 - 6x - 23$
   (b) $x + 3y + 6x + 4y$
   (c) $6xy + 13x - 2yx - 5x$
   (d) $6x - 20y + 7z - 8x + 25y - 11z$

6. Find the sum of each of the following pairs of expressions.
   (a) $2x + 4y$ and $-5y$
   (b) $-b - 4a$ and $7b - 6a$
   (c) $6d - 4c$ and $-7c + 6d$
   (d) $3pq - 6hk$ and $-3qp + 14kh$

7. Write down an algebraic expression for each of the following statements.
   (a) Subtract the cube root of the product of h and $3k$ from the square of the sum of p and q.
   (b) The total value of x 20-cent coins and y 5-dollar notes in cents.

8. Given that $a = 3$, $b = -4$ and $c = -2$, evaluate each of the following expressions.
   (a) $\frac{3a-b}{2c} + \frac{3a-c}{c-b}$
   (b) $\frac{2c-a}{3c+b} - \frac{5a+4c}{c-a}$
   (c) $\frac{a+b+2c}{3c-a-b} - \frac{5c}{4b}$
   (d) $\frac{b-c}{3c+4b} + \left(\frac{bc}{a} + \frac{ac}{b}\right)$

9. Simplify each of the following.
   (a) $137x + (-24x)$
   (b) $76x + (-183x)$
   (c) $-73x + 26x$
   (d) $-95x + 113x$
   (e) $-84y - 23y$
   (f) $-714y - 716y$
   (g) $-59z - (-48z)$
   (h) $-34z - (-191z)$

10. Simplify each of the following expressions.
    (a) $15a + (-7b) + (-18a) + 4b$
    (b) $-3h + (-5k) - (-10k) - 7h$
    (c) $9p - (-2q) - 8p - (-12q)$
    (d) $-7x - (-15y) - (-2x) + (-6y)$
    (e) $29x + (-13) - 7x - (-18y) + (-20) - 15y$
    (f) $-6y - 14 - (-8yz) - 6z + 23 + 9y - 8z + 2zy$

11. (i) Simplify the expression $3p + (-q) - 7r - (-8p) - q + 2r$.
    (ii) Find the value of the expression when $p = 2$, $q = -1\frac{1}{2}$ and $r = -5$.

12. Raju is $12m$ years old. His son was born when he was $9m$ years old.
    (i) Find Raju's age 5 years later.
    (ii) Find the sum of their ages in 5 years' time.

13. Li Ting bought 8 books at $\$w$ each and 7 pens at $\$m$ each. She had $\$(3w + 5m)$ left. Find the amount of money she had at first.

Basic | Intermediate

Exercise 6A

11. At a famous pizza shop, for every two people who order cheese pizza, there are five people who order chicken pizza.
 (a) If a people order cheese pizza, how many people order chicken pizza?
 (b) If b people order chicken pizza, how many people order cheese pizza?
 (c) If there are a total of c people in the shop, how many of them order cheese pizza?

6.3 Expansion and factorisation of linear expressions

In this section, we will learn what expressions such as $2(x + y)$, $-(x + y)$ and $-3(x - y)$ mean.

A. Expansion of linear expressions

What does $2(x + y)$ mean?

Applying what we know about whole numbers, the product 2×3 means 2 groups of 3, and it can be represented by number discs as shown in Fig. 6.8, i.e. $2 \times 3 = 6$.

Fig. 6.8

Thus, $2(x + y)$ represents $2 \times (x + y)$, just like how the notation $2x$ means $2 \times x$.
So $2(x + y)$ means 2 groups of $(x + y)$, and it can be represented by algebra discs as shown in Fig. 6.9, i.e. $2(x + y) = 2x + 2y$.

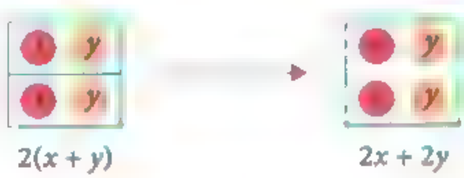


Fig. 6.9

> **Attention**
> Fig. 6.9 shows that $2(x + y) \neq 2x + y$, i.e. we have to multiply every term within the brackets by 2.

What we have just done is to *expand* $2(x + y)$ to become $2x + 2y$.
Expansion is the process of expressing an algebraic expression as the *sum* and/or *difference* of two or more terms. The process of *expansion* is different from simplification because it is debatable which of the two expressions, $2(x + y)$ or $2x + 2y$, is simpler.

> **Attention**
> $2x$ and $2y$ are called the *terms* of the expression $2x + 2y$.
> 2 and x can be called *factors* of $2x$. Similarly, 2 and $(x + y)$ are factors of $2(x + y)$.

What does $-(x + y)$ mean?

To recap what we have learnt in Chapter 2, the negative of 1 can be obtained by flipping the 1 disc to obtain -1:

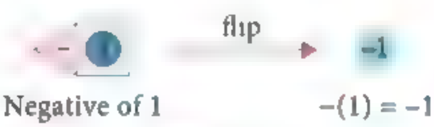


Fig. 6.10

Similarly, the negative of (-1), i.e. $-(-1)$, can be obtained by flipping the -1 disc to obtain 1:

Fig. 6.11

Thus, $-(x + y)$ means the negative of $(x + y)$, and it can be represented by algebra discs as shown in Fig. 6.12, i.e. $-(x + y) = -x + (-y) = -x - y$.

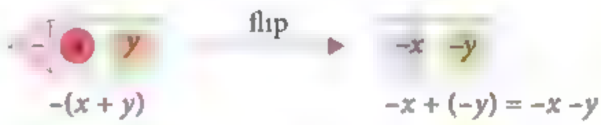


Fig. 6.12

Attention

Fig. 6.12 shows that $-(x + y) \neq -x + y$, i.e. we have to multiply every term within the brackets by -1.

What does $-2(x + y)$ mean?

$-2(x + y)$ means the negative of '2 groups of $(x + y)$', and it can be represented by algebra discs as shown in Fig. 6.13, i.e. $-2(x + y) = -2x + (-2y) = -2x - 2y$.

Fig. 6.13

Attention

Fig. 6.13 shows that $-2(x + y) \neq -2x + y$, i.e. we have to multiply every term within the brackets by -2.

Expansion of linear expressions

Work out the answers to each of the following, using algebra discs as shown above. Write down the final answer for each part.

(a) $3(x + y)$ **(b)** $3(x - y)$ **(c)** $-(x - y)$ **(d)** $-3(x - y)$

(e) $3(-x + 2y)$ **(f)** $-3(-x + 2y)$ **(g)** $3(-x - 2)$ **(h)** $-3(-x - 2)$

Hint: For part (b), $3(x - y) = 3[x + (-y)]$. Why?

From the Investigation on page 145, we observe the following:

The Distributive Law

$$a(b + c) = ab + ac$$

This is called the Distributive Law because the first factor a is *distributed*, or multiplied separately, to each of the two terms, b and c, in the second factor $(b + c)$. In particular,

Negative of $(x + y)$ and negative of $(x - y)$

$$-(x + y) = -x - y \quad \text{and} \quad -(x - y) = -x + y$$

Two expressions are equivalent if the value of both expressions is the same for *any* value we substitute into the variables, e.g. $a(b + c) = ab + ac$ for any values of a, b and c. Writing $a(b + c)$ in its equivalent form $ab + ac$ can help us to simplify expressions such as those in Worked Example 5.

The Distributive Law can be visualised using the rectangle $PQRS$ (see Fig 6.14).

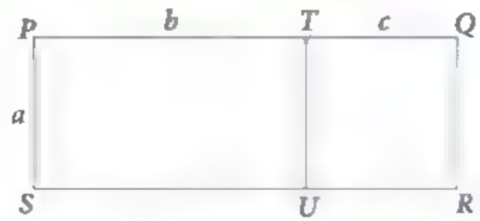


Fig. 6.14

Since area of rectangle $PQRS$ = area of rectangle $PTUS$ + area of rectangle $TQRU$,

then $a(b + c) = ab + ac$,

which is the Distributive Law.

4

Expanding expressions using Distributive Law

Expand each of the following expressions.

(a) $6(x - 2)$ **(b)** $-3(4x - y)$ **(c)** $7 - a(-5x + z)$

Solution

(a) $6(x - 2) = 6x - 12$

(b) $-3(4x - y) = -12x + 3y$ $\quad -3 \times (-y) = +3y$

$= 3y - 12x$ $\quad$ rearrange order of terms

(c) $7 - a(-5x + z) = 7 + 5ax - az$

Problem-solving Tip

(b) We usually leave our answer as $3y - 12x$, which looks simpler than $-12x + 3y$.

Similar and

Exercise 6B
Questions 1(a)–(h), 2

Expand each of the following expressions.

(a) $2(x - 7)$ **(b)** $-5(3x - 4y)$

(c) $8 - a(-x + 2z)$ **(d)** $6 - b(-3y - z)$

Expanding and simplifying expressions

Expand and simplify each of the following expressions.

(a) $-3(x + 2y) - 4(x - y)$ **(b)** $5y - 2[3z - 2(y + 3z)]$

(a) $-3(x + 2y) - 4(x - y)$

$= -3x - 6y - 4x + 4y$ Distributive Law

$= -3x - 4x + 4y - 6y$ group like terms

$= -7x - 2y$ recall: $-5x - 2x$

(b) $5y - 2[3z - 2(y + 3z)]$

$= 5y - 2(3z - 2y - 6z)$ Distributive Law for innermost brackets

$= 5y - 2(3z - 6z - 2y)$ group like terms

$= 5y - 2(-3z - 2y)$ recall: $2x - 5x = -3x$

$= 5y + 6z + 4y$ Distributive Law

$= 5y + 4y + 6z$ group like terms

$= 9y + 6z$

Problem-solving Tip

(b) Simplify the expression in innermost brackets first.

Expand and simplify each of the following expressions.

(a) $6(4x + y) + 2(x - y)$

(b) $x - \{y - 3(2x - y)\}$

(c) $7x - 2[3(x - 2) - 2(x - 5)]$

B. Factorisation of linear expressions

In Section 6.3A, we have learnt how to expand linear expressions, e.g. $2(x + y) = 2x + 2y$.

Now, let us learn how to carry out the *reverse* process, e.g. $2x + 2y = 2(x + y)$.

This process is called *factorisation* because we *factorise* $2x + 2y$ into its two *factors*, 2 and $(x + y)$, to become $2(x + y)$.

Fig. 6.15 illustrates the processes of expansion and factorisation as the reverse of each other.

$$2(x + y) \underset{\text{factorise}}{\overset{\text{expand}}{\rightleftarrows}} 2x + 2y$$

Fig. 6.15

Information

This diagram helps to depict the reverse process of expansion and factorisation.

> **Factorisation of algebraic expressions**
>
> **Factorisation** is the process of expressing an algebraic expression as the *product* of two or more factors. It is the reverse of expansion.

Extracting positive common factors

To factorise $2x + 2y$, we need to identify common factor(s) of the two terms $2x$ and $2y$. In this case, the common factor is 2. Then we *extract* the common factor 2 from both terms, $2x$ and $2y$, to obtain $2(x + y)$.

Sometimes, there is more than one common factor, e.g. in $6xy - 9x$, the common factors are 3 (which is the HCF of 6 and 9) and x. We extract $3x$ from both terms, $6xy$ and $9x$, to obtain $3x(2y - 3)$.

The factorisation is *incomplete* if we only extract one common factor, e.g. $3(2xy - 3x)$.

Worked Example 6

Factorisation by extracting positive common factors

Factorise each of the following expressions completely.

(a) $8x + 12$ **(b)** $-24az + 4ay$ **(c)** $-20b + 15by - 10bz$

Solution

(a) $8x + 12 = 4(2x + 3)$ HCF of 8 and 12 = 4

(b) $-24az + 4ay = 4ay - 24az$ rearrange order of terms
$= 4a(y - 6z)$ common factors are 4 and a

(c) $-20b + 15by - 10bz$
$= 15by - 20b - 10bz$ rearrange order of terms
$= 5b(3y - 4 - 2z)$ common factors are 5 and b

> **Attention**
> (c) You can also leave your answer as $5b(3y - 2z - 4)$

Exercise 6B

Factorise each of the following expressions completely.

(a) $10x + 25$ **(b)** $18 - 12x$
(c) $21a - 14ay$ **(d)** $33ax + 27a + 3ay$
(e) $18x - 54xy + 36xz$ **(f)** $-10x + 30xy - 50xz$

Extracting negative common factors

In Section 6.3A, we learnt how to expand $-(x + y)$ to get $-x - y$.

Now, we are going to do the reverse by factorising $-x - y$ to get $-(x + y)$.

> **Extracting negative common factor**
>
> $$-x - y = -(x + y)$$

Observe that when we extract the negative sign, i.e. factor out the common factor -1, we have to change the sign inside the brackets as shown above.

However, for $-x + y$, there is no need to extract the negative sign because we can write it as $y - x$ (see Worked Example 6(**b**)).

Factorisation by extracting negative common factors

Factorise each of the following expressions completely.

(a) $-6x - 15$ (b) $-6a - 24az - 4ay$

(a) $-6x - 15 = \overset{\text{factor out } -3}{-3(2x + 5)}$ HCF of 6 and 15 = 3, so -3 is a common factor

change sign

(b) $-6a - 24az - 4ay = \overset{\text{factor out } -2a}{-2a(3 + 12z + 2y)}$ HCF of 6, 24 and 4 = 2

change sign

Factorise each of the following expressions completely.

(a) $-5h - 25$ (b) $-14ay - 21a$

(c) $-9z - 24bz - 15cz$ (d) $-4 - 16x - 20xy$

Introductory Problem Revisited

Now that we know how to add, subtract, expand, factorise and simplify linear expressions, we will be able to solve the Introductory Problem.

How can we represent a two-digit positive integer using letters?
Let the tens digit be x and the ones digit be y.
Then the number is $10x + y$, where x and y are integers such that $1 \leq x \leq 9$ and $0 \leq y \leq 9$.
For example, if the number is 37, then $x = 3$ and $y = 7$,
so $10x + y = 10 \times 3 + 7 = 30 + 7 = 37$.
Notice that $10x = 10 \times 3 = 30$ because $x = 3$ is in the tens place.
Why is it that x cannot be equal to 0 but y can be 0?

Attention

We cannot write the number as xy because xy means $x \times y$. For example, if the number is 37, then $x = 3$ and $y = 7$, but $xy = 3 \times 7 = 21$ is not the correct number.

Now, let us revisit the game.

Step 1: Think of a two-digit positive integer that is greater than 20.

(a) Let the tens digit of the two-digit positive integer that is greater than 20 be x and the ones digit be y. What values can x and y take?

(b) Write down an expression for the two-digit number in terms of x and y.

Step 2: Add the two digits together.

(c) Write down an expression for the sum of the two digits.

Subtract the sum of the two digits from your original number to get a new number.

(d) Write down an expression for this new number. What properties can you observe about this expression?

Add the two digits of this new number together to obtain the final number.

(e) We will not use letters to represent this final number. Instead, based on the properties that you have observed in part (d), what can you say about the value of this final number? Why?

Do you see the power of algebra in solving problems like this?

1. Do I understand why the expansion of $a(b + c) = ab + ac$ and *not* $ab + c$?
2. What do I understand about the relationship between the expansion and factorisation of algebraic expressions?
3. How can I check if I have expanded or factorised an algebraic expression correctly?

Basic | Intermediate

Exercise

1. Expand each of the following expressions.

(a) $-(x + 5)$ (b) $-(4 - x)$
(c) $2(3y + 7)$ (d) $8(2y - 5)$
(e) $-8(3a - 4b)$ (f) $2a(x - y)$
(g) $5 - b(-6w + 2x)$ (h) $13 + 3c(-y - 3z)$

2. Ken is x years old. Ken's uncle is now four times as old as Ken will be in 5 years. Find the age of Ken's uncle now.

3. Expand and simplify each of the following expressions.

(a) $5(a + 2b) - 3b$
(b) $7(p + 10q) + 2(6p + 7q)$
(c) $a + 3b - (5a - 4b)$
(d) $x + 3(2x - 3y + z) + 7z$

4. A pear costs x cents. An orange costs y cents less than a pear. Find the cost of 4 pears and half a dozen oranges.

5. Factorise each of the following expressions completely.

(a) $12x + 9$ (b) $27b - 36by$
(c) $-16pq + 6pr$ (d) $8ax + 12a - 4az$
(e) $-6h + 30hk - 24hn$ (f) $-4mx - 6my + 18mz$

6. Factorise each of the following expressions completely.

(a) $-25y - 35$ (b) $-64ax - 16ay$
(c) $-81w - 9wx - 18wy$ (d) $-24x - 36 - 12xz$

7. Subtract

(a) $-6x - 3$ from $2x - 5$,
(b) $6x - y + 5z$ from $10x - 2y + z$,
(c) $8p + 9q - 5rs$ from $-4p - 4q + 15sr$,
(d) $8a - 3b + 5c - 4d$ from $10a - b - 4c - 8d$.

Exercise 6B

8. Expand and simplify each of the following expressions.
   (a) $4u - 3(2u - 5v)$
   (b) $-2a - 3(a - b)$
   (c) $7m - 2n - 2(3n - 2m)$
   (d) $5(2x + 4) - 3(-6 - x)$
   (e) $-4(a - 3b) - 5(a - 3b)$
   (f) $5(3p - 2q) - 2(3p + 2q)$
   (g) $x + y - 2(3x - 4y + 3)$
   (h) $3(p - 2q) - 4(2p - 3q - 5)$
   (i) $9(2a + 4b - 7c) - 4(b - c) - 7(-c - 4b)$
   (j) $-4\{5(2x + 3y) - 4(x + 2y)\}$

9. Expand and simplify each of the following expressions.
   (a) $-2\{3a - 4[a - (2 + a)]\}$
   (b) $5\{3c - [d - 2(c + d)]\}$

10. Subtract
   (a) $14x^2 - 12x - 6$ from the sum of $2x^2 - 3x + 11$ and $7x^2 - 3x - 4$,
   (b) $2x^3 + 7x^2 - 4x - 13$ from the sum of $17x^3 - 12x + 11$ and $x^3 + 5$,
   (c) the sum of $w^2 - 10w + 25$ and $3w^2 - 7w$ from the product of 4 and $2 - 5w$,
   (d) the product of 3 and $6w^2 - 7w + 1$ from the product of -2 and $-3 - 7w$.

11. Factorise each of the following expressions completely.
   (a) $5x + 10x(b + c)$
   (b) $3xy - 6x(y - z)$
   (c) $2x(7 + y) - 14x(y + 2)$
   (d) $-39b^2 - 13ab$
   (e) $-3a(2 + b) - 18a(b + 1)$
   (f) $(-4xy^2 - 16xy)[a(3y + 2) - 2a(y - 1)]$

6.4 Linear expressions with fractional coefficients

In previous sections, we have only encountered linear expressions with integer coefficients. In this section, we will work with fractional coefficients.

What is the difference between $\frac{n}{2}$ and $\frac{1}{2}n$? Are they equal? How about $\frac{n+1}{3}$ and $\frac{1}{3}(n+1)$? Are they equal?

Comparing algebraic notations

1. You can use the same spreadsheet (see Table 6.3) in Section 6.1D Investigation on page 133.
Type the expressions, as shown in Table 6.7, in the cells I1, J1, K1 and L1, and key in the formulae in the cells I2, J2, K2 and L2 (/ and * mean ÷ and × in the spreadsheet respectively).

	A	I	J	K	L
1	n	$n/2$	$(1/2)n$	$(n+1)/3$	$(1/3)(n+1)$
2	1	=A2/2	=(1/2)*A2	=(A2+1)/3	=(1/3)*(A2+1)
3	2				
4	3				
5	4				
6	5				

Table 6.7

You should obtain the following results after you have entered the above formulae in the cells. (Do not type the values in I2 to L2 directly!)

	A	I	J	K	L
1	n	$n/2$	$(1/2)n$	$(n+1)/3$	$(1/3)(n+1)$
2	1	0.5	0.5	0.666666667	0.666666667
3	2				
4	3				
5	4				
6	5				

Table 6.8

2. The cell I2 shows the number 0.5 because when $n = 1$ (in the cell A2), according to the formula A2/2,
$\frac{n}{2} = n / 2 = n \div 2 = 1 \div 2 = 0.5$.
Explain how the spreadsheet obtained the numbers in the cells J2 to L2.

3. Select the cells I2 to L2. To copy the respective formulae in I2 to L2 down the rows, move the cursor to the bottom right corner of the cell L2 until a '+' appears. Then click on the '+' and drag downwards to L6. You should obtain the following results for rows 3 and 4.

	A	I	J	K	L
1	n	$n/2$	$(1/2)n$	$(n+1)/3$	$(1/3)(n+1)$
2	1	0.5	0.5	0.666666667	0.666666667
3	2	1	1	1	1
4	3	1.5	1.5	1.333333333	1.333333333
5	4				
6	5				

Table 6.9

Copy and complete Table 6.9 for Rows 5 and 6.

4. Compare the values in Columns I to L and answer the following questions.

   (i) Is $\frac{n}{2} = \frac{1}{2}n$? Why?

   (ii) Is $\frac{n+1}{3} = \frac{1}{3}(n+1)$? Why?

From the above Investigation, we obtain the following results which are true in general for any n.

Meanings of more algebraic notations

- $\frac{n}{2} \quad \frac{1}{2}n$
- $\frac{n+1}{3} = \frac{1}{3}(n+1)$

Dividing an expression by a number m is equivalent to multiplying the same expression by the reciprocal of the number, i.e. multiplying by $\frac{1}{m}$.

We have learnt in Section 6.1G that a linear expression in one variable x is an algebraic expression that contains only one term in x, with or without a constant term.

Given examples include $-\frac{1}{3}x$, which has a **fractional coefficient**, $-\frac{1}{3}$.

Other examples of *linear expressions* with fractional coefficients are: $\frac{1}{2}x - 6$, $-\frac{5}{3}x + \frac{7}{4}$ and $\frac{1}{2}x + \frac{3}{4}y$.

$\frac{1}{3}(x+1)$ and $\frac{x+1}{3}$ are also linear expressions with fractional coefficients, because $\frac{x+1}{3} = \frac{1}{3}(x+1) = \frac{1}{3}x + \frac{1}{3}$.

To simplify linear expressions with fractional coefficients, we group like terms together and convert the fractional coefficients to equivalent fractions with the same denominator. Worked Example 8 illustrates this.

8

Simplifying linear expressions with fractional coefficients

Simplify each of the following expressions.

(a) $\frac{1}{2}x - \frac{1}{9}y - \frac{1}{8}x + \frac{11}{3}y$ (b) $\frac{2}{3}\left[2x - 5(x - 6y)\right]$

Solution

(a) $\frac{1}{2}x - \frac{1}{9}y - \frac{1}{8}x + \frac{11}{3}y$

$= \frac{1}{2}x - \frac{1}{8}x - \frac{1}{9}y + \frac{11}{3}y$ group like terms

$= \frac{4}{8}x - \frac{1}{8}x - \frac{1}{9}y + \frac{33}{9}y$ convert to equivalent fractions: $\frac{1}{2} = \frac{4}{8}$ and $\frac{11}{3}$

$= \frac{3}{8}x + \frac{32}{9}y$

(b) $\frac{2}{3}[2x-5(x-6y)]$

$= \frac{2}{3}(2x-5x+30y)$ Distributive Law for innermost brackets

$= \frac{2}{3}(-3x+30y)$ recall: $2x - 5x = -3x$

$= \frac{2}{3}\times(-3x)+\frac{2}{3}\times 30y$

$= -2x + 20y$

$= 20y - 2x$

Further Questions
Exercise 6C
Questions 1(a)–(d), 2(a)–(d), 4(a), (b)

Simplify each of the following expressions.

(a) $\frac{1}{2}x+\frac{1}{4}y-\frac{2}{5}y-\frac{1}{3}x$ (b) $\frac{1}{8}[-y-3(16x-3y)]$

9

coefficients

Express each of the following as a fraction in its simplest form.

(a) $\frac{1}{3}x+\frac{2x-5}{7}$ (b) $\frac{2x-5}{3}-\frac{1}{5}(3x-2)$

(a) $\frac{1}{3}x+\frac{2x-5}{7}$

$= \frac{x}{3}+\frac{2x-5}{7}$

$= \frac{7x}{21}+\frac{3(2x-5)}{21}$ convert to equivalent fractions

$= \frac{7x+3(2x-5)}{21}$

$= \frac{7x+6x-15}{21}$ Distributive Law

$= \frac{13x-15}{21}$

(b) $\frac{2x-5}{3}-\frac{1}{5}(3x-2)$

$= \frac{2x-5}{3}-\frac{3x-2}{5}$

$= \frac{5(2x-5)}{15}-\frac{3(3x-2)}{15}$ convert to equivalent fractions

$= \frac{5(2x-5)-3(3x-2)}{15}$ combine into single fraction

$= \frac{10x-25-9x+6}{15}$ Distributive Law

$= \frac{10x-9x-25+6}{15}$ group like terms

$= \frac{x-19}{15}$

Problem-solving Tip

Always combine the terms into a single fraction before removing the brackets.

Attention

You can leave your answer for **(a)** as $\frac{13x-15}{21}$ or $\frac{1}{21}(13x-15)$. Similarly for **(b)** as well.

(b) $-(x-y) = -x+y$

1. Express each of the following as a fraction in its simplest form.

   (a) $\frac{1}{2}(x-3)+\frac{2x-5}{3}$ (b) $\frac{x-2}{4}-\frac{2x-7}{3}$

2. Express each of the following as a fraction in its simplest form.

   (a) $\frac{x-1}{3}+\frac{1}{2}-\frac{1}{4}(2x-3)$ (b) $2x+\frac{x-4}{9}-\frac{2x-5}{3}$

Work in pairs.

Five pairs of equivalent expressions can be found in Table 6.10. An example of a pair of equivalent expressions is $3x - 12$ and $3(x - 4)$. Match and justify each pair of equivalent expressions.

A $\frac{1-x}{6}$	**B** $\frac{-23x+75}{12}$	**C** $7(ay-7y)$	**D** $3(x-2y)-2(3x-y)$	**E** $\frac{x-3}{2}-\frac{2x-5}{3}$
F $-3x-4y$	**G** $\frac{3(x+3)}{4}-\frac{4(2x+3)}{3}$	**H** $\frac{-x-19}{6}$	**I** $29x-3y$	**J** $7y(a-7)$
K $7ay-49y$	**L** $-25x-9y$	**M** $2x-3[5x-y-2(7x-y)]$	**N** $\frac{-23x-21}{12}$	**O** $-3x-8y$

Table 6.10

1. What do I already know about the addition and subtraction of fractions that could help me simplify linear algebraic expressions with fractional coefficients?
2. How can I check if an algebraic expression has been simplified correctly?
3. What have I learnt in this section or chapter that I am still unclear of?

Exercise 6C

1. Simplify each of the following expressions.
   (a) $\frac{1}{4}x+\frac{1}{5}y-\frac{1}{6}x-\frac{1}{10}y$
   (b) $\frac{2}{3}a-\frac{1}{7}b+2a-\frac{3}{5}b$
   (c) $\frac{5}{9}c+\frac{3}{4}d-\frac{7}{8}c-\frac{4}{3}d$
   (d) $2f-\frac{5}{3}h+\frac{9}{4}k-\frac{1}{2}f-\frac{28}{5}k+\frac{5}{4}h$

2. Simplify each of the following expressions.
   (a) $5a+4b-3c-\left(2a-\frac{3}{2}b+\frac{3}{2}c\right)$
   (b) $\frac{1}{2}[2x+2(x-3)]$
   (c) $\frac{2}{5}[12p-(5+2p)]$
   (d) $\frac{1}{2}[8x+10-6(1-4x)]$

3. Express each of the following as a fraction in its simplest form.
   (a) $\frac{1}{2}x+\frac{2x}{5}$
   (b) $\frac{a}{3}-\frac{1}{4}a$
   (c) $\frac{2h}{7}+\frac{1}{5}(h+1)$
   (d) $\frac{3x}{8}-\frac{1}{4}(x+2)$
   (e) $\frac{4x+1}{5}+\frac{3x-1}{2}$
   (f) $\frac{3y-1}{4}-\frac{2y-3}{6}$
   (g) $\frac{1}{4}(a-2)-\frac{a+7}{8}$
   (h) $\frac{1}{3}(3p-2q)-\frac{4p-5q}{4}$

4. Simplify each of the following expressions.
   (a) $y-\frac{2}{3}(9x-3y)$
   (b) $-\frac{1}{3}\{6(p+q)-3[p-2(p-3q)]\}$

5. Express each of the following as a fraction in its simplest form.
   (a) $\frac{7(x+3)}{2}+\frac{5(2x-5)}{3}$
   (b) $\frac{3x-4}{5}-\frac{3(x-1)}{2}$
   (c) $\frac{3}{4}(z-2)-\frac{4(2z-3)}{5}$
   (d) $\frac{2(p-4q)}{3}-\frac{3(2p+q)}{2}$
   (e) $-\frac{2b}{3}-\frac{3(a-2b)}{5}$
   (f) $\frac{2}{5}(x+3)-\frac{1}{2}+\frac{3x-4}{4}$
   (g) $\frac{a+1}{2}-\frac{a+3}{3}-\frac{5a-2}{4}$
   (h) $\frac{x+1}{2}+\frac{x+3}{3}-\frac{5x-1}{6}$
   (i) $\frac{2(a-b)}{7}-\frac{2a+3b}{14}+\frac{a+b}{2}$
   (j) $\frac{x+3}{3}+\frac{5}{6}(3x+4)+1$

6. Express each of the following as a fraction in its simplest form.
   (a) $\frac{5(p-q)}{2}-\frac{2q-p}{14}-\frac{2(p+q)}{7}$
   (b) $\frac{2a+b}{3}-\left[\frac{3(a-3b)}{2}-\frac{4(a+2b)}{5}\right]$
   (c) $\frac{3(f-h)}{4}-\frac{7(h+k)}{6}+\frac{5(k-f)}{2}$
   (d) $4-\frac{x-y}{3}-\frac{3(y+4z)}{4}+\frac{5}{8}(x+3z)$

In this chapter, we embark on the exciting journey of learning algebra. First, we see algebra as a means to express relationships between quantities. Algebra is a form of generalised arithmetic that uses **notations** to help us work with abstract ideas about numbers. For example, by writing $a + b = b + a$, we can express the idea that changing the order in which two numbers are added will not change the sum. The language of algebra is highly applicable in mathematics and provides us with the tools to **model** real-world situations. Algebraic manipulations are based on the idea of **equivalence**, where the values of variables remain the same even as an expression is changed

Summary

1. In algebra, **letters** are used to represent *different kinds of numbers*, such as specific unknown numbers, generalised numbers and variables. Examples of algebraic notations:

$$2n = 2 \times n = n + n \qquad (2n \neq 2 + n)$$
$$n^2 = n \times n \qquad (n^2 \neq 2n)$$
$$(2n)^2 = 4n^2 \qquad ((2n)^2 \neq 2n^2)$$
$$\frac{n}{2} = \frac{1}{2}n$$
$$\frac{n+1}{3} = \frac{1}{3}(n+1)$$

- Give two other examples of algebraic notations and explain what they mean.

2. A **linear expression** in one *variable* x is an algebraic expression that contains only one term in x, with or without a constant term.
An example of a linear expression in x is $2x + 7$: it consists of two *terms*, $2x$ and 7, where the *coefficient* of x is 2 and the *constant term* is 7.
Another example of a linear expression in x is $-\frac{1}{3}x$: it has a *fractional coefficient* $-\frac{1}{3}$ and no constant term (or the constant term is 0).
Examples of linear expressions in two variables x and y are $4x + y - 8$ and $\frac{1}{2}x + \frac{3}{4}y$.
- Give another example of a linear expression in x with integer coefficients.
- Give another example of a linear expression in x with fractional coefficients.
- Give two other examples of linear expressions in two variables x and y.

3. **Evaluating** an algebraic expression is the process of finding the value of the expression when its variables are given certain values.

4. **Addition and subtraction of like terms**

To add or subtract like terms, we add or subtract their coefficients, which will be the coefficient of the answer. The variable remains the same. The observations are the same as those for the addition and subtraction of real numbers.

E.g. $(-2) + (-3) = -5$;	likewise, $(-2x) + (-3x) = -5x$.
E.g. $5 + (-2) = 5 - 2 = 3$;	likewise, $5x + (-2x) = 5x - 2x = 3x$;
and $-2 + 5 = 5 - 2 = 3$;	and $-2x + 5x = 5x - 2x = 3x$.
E.g. $2 - 5 = -3$;	likewise, $2x - 5x = -3x$.
E.g. $-5 - 2 = -7$;	likewise, $-5x - 2x = -7x$.
E.g. $5 - (-2) = 5 + 2 = 7$;	likewise, $5x - (-2x) = 5x + 2x = 7x$.

We *cannot further simplify* the addition or subtraction of *unlike terms*.

5. **Factorisation** is the process of expressing an algebraic expression as the *product* of two or more factors. It is the reverse of expansion.

$$\begin{tikzcd} 2(x+y) \arrow[r, bend left, "\text{expand}"] & 2x+2y \arrow[l, bend left, "\text{factorise}"] \end{tikzcd}$$

6. **The Distributive Law**

$a\,(b + c) = ab + ac$

7. **Negative of $(x + y)$ and negative of $(x - y)$**

$-(x + y) = -x - y$ and $-(x - y) = -x + y$

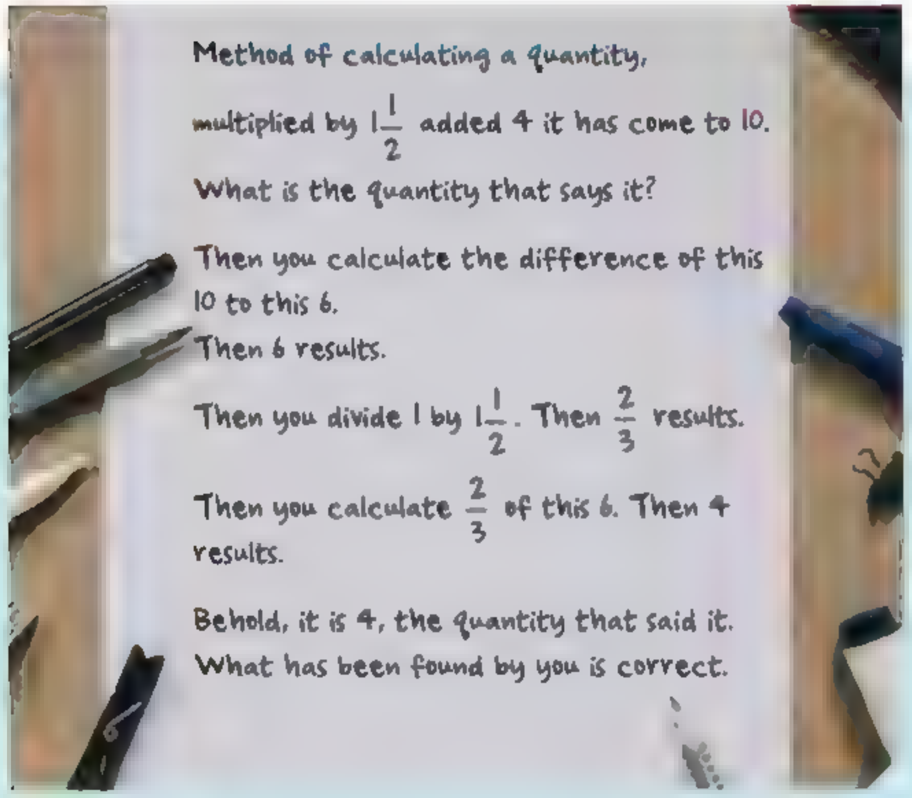
8. **Extracting negative common factor**

factor out -1

$-x - y = -(x + y)$

CHAPTER 7

Linear Equations



In this chapter, we are going to continue learning the basics of algebra. Specifically, we will learn to solve linear equations. Before the invention of symbolic algebra, people used algorithms to find the unknowns of algebraic equations. For example, the ancient Egyptians were able to solve questions involving a single variable. On the left is an example of such a problem that has been translated into English.

Can you tell what the person was trying to do?

In this chapter, we will further explore the power of algebraic notations in solving equations.

Learning Outcomes

What will we learn in this chapter?

- What linear equations and mathematical formulae are
- How to solve linear equations with integer or fractional coefficients, and fractional equations that can be converted into linear equations with integer coefficients
- How to evaluate an unknown in a formula
- Why linear equations have useful applications in real-world contexts

Introductory Problem

Ken has some sweets to share with his cousins.
If he gives them 6 sweets each, he would have 5 sweets left.
If he gives them 7 sweets each, he would be short of 4 sweets.
How many cousins does Ken have?
Can you solve this problem by drawing models or by using some form of reasoning?

It is not easy to use models or reasoning to solve the Introductory Problem.
In this chapter, we will learn how to use algebra to solve such problems.

7.1 Linear equations

A. Concept of equation

In Chapter 6, we learnt about *algebraic expressions*.

For example, if your friend has x cupcakes and you have 2 more cupcakes than your friend, we can write an algebraic expression to represent the number of cupcakes you have, i.e. $x + 2$.
If we also know that you have 5 cupcakes, we will be able to write an algebraic **equation**,

$$x + 2 = 5$$

The value on the left-hand side (LHS) of the equation is *equal to* the value on the right-hand side (RHS).

There is a particular value of x for which LHS = RHS. In this case, it is

$$x = 3$$

This is the **solution** of the equation.

Now if $x = 4$, LHS $= x + 2 = 4 + 2 = 6$, and LHS $\neq$ RHS.

Thus, $x = 3$ *satisfies* the equation, but $x = 4$ does not.

Notice that the equation $x + 2 = 5$ involves only the linear expression $x + 2$. Such an algebraic equation, with only linear expressions on either side of the equation is called a **linear equation**.

1. Some expressions and equations are shown below. Identify which are expressions and which are equations. Then explain the difference between an expression and an equation.
   (a) $x + 2$ (b) $x + 2 = 5$ (c) $6(y - 1)$ (d) $6(y - 1) = 0$
   (e) $5x + 4 = 2x - 8$ (f) $x^2 - 2 = 0$ (g) $3x + y = -7$ (h) $9\sqrt{x} - xy = 3$

2. Which of the above equations are linear equations? Explain.

Some linear equations are in *one variable*, e.g. $x + 2 = 5$, while others are in two variables, e.g. $3x + y = -7$. In this chapter, we will only learn how to solve linear equations in one variable.

B. Solving linear equations

The '=' sign in an equation means that the expression on the LHS of the equation must have the same value as the expression on the RHS of the equation. We will explore this concept further using the idea of a balancing scale (or "balance" for short).

In Fig. 7.1, the the number on the left pan is equal to the number on the right pan, i.e. $2 + 3 = 5$. The balancing scale is *balanced*.

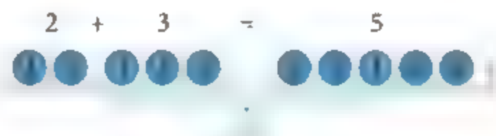


Fig. 7.1

If we add 1 to the numbers on the left pan, then the balance will be *tilted* because the numbers on the left and right pans are not equal, i.e. $2 + 3 + 1 \neq 5$ (see Fig. 7.2(**a**)).

To *maintain the balance*, we need to add 1 to the number on the right pan so $2 + 3 + 1 = 5 + 1$ (see Fig. 7.2(**b**)).

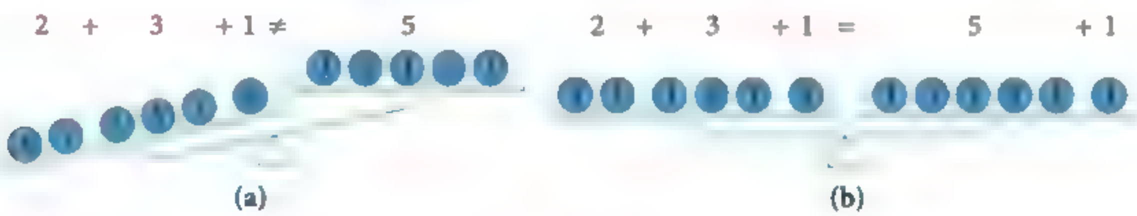


Fig. 7.2

Likewise, we need to subtract the same number from the numbers on both pans, or multiply or divide the numbers on both pans by the same number, to maintain the balance.
We will now explore how to solve linear equations.

Part 1: $x + b = d$, where b and d are constants

Example: $x + 2 = 5$

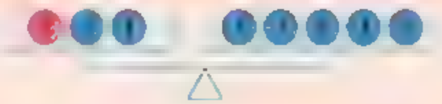
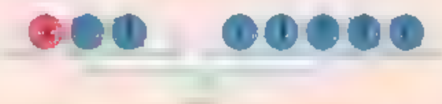
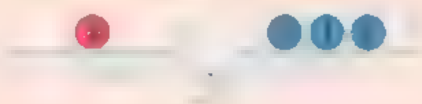
Pictorial representation	Algebraic representation
	$x + 2 = 5$
Take away 2 discs from each of the two pans: 	Subtract 2 from both sides of the equation: $x + 2 - 2 = 5 - 2$
	Simplify both sides of the equation: $x = 3$

Table 7.1

1. Solve each of the following equations using the idea of the balancing scale.
   (a) $x + 3 = 7$ (b) $x - 4 = 6$ (c) $x + 2 = -5$ (d) $x - 8 = -1$

When no discs can be taken away, what can we add to both sides of the equation to obtain zero pairs?

Part 2: $ax + b = d$, where a, b and d are constants

Example: $2x - 3 = 5$

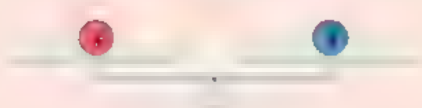
Pictorial representation	Algebraic representation
Since LHS = $2x - 3 = 2x + (-3)$, we can represent the equation as follows: 	$2x - 3 = 5$
Add to each of the two pans: zero pair 	Add 3 to both sides of the equation. $2x - 3 + 3 = 5 + 3$
	Simplify both sides of the equation: $2x = 8$
	Divide both sides of the equation by 2: $\frac{2x}{2} = \frac{8}{2}$
	Simplify both sides of the equation: $x = 4$

Table 7.2

2. Solve each of the following equations using the idea of the balancing scale.
(a) $2x - 5 = 7$ (b) $3x + 8 = 2$ (c) $-4x + 3 = -5$ (d) $-x - 8 = -8$
For part (c), you can divide by a negative number at an appropriate step.

3. For the equation $2x - 5 = 7$ in Question 2(a), can you divide both sides of the equation by 2 straightaway? Explain.

Part 3: $ax + b = cx + d$, where a, b, c and d are constants

Example: $5x + 4 = 2x - 8$

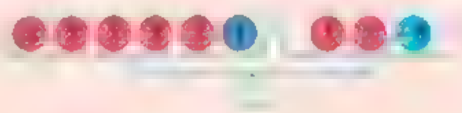
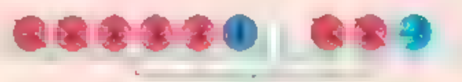
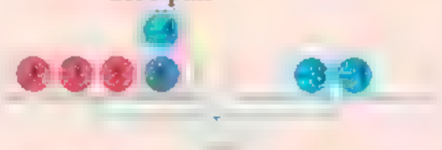
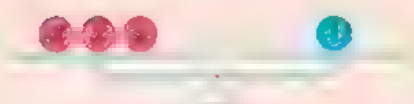
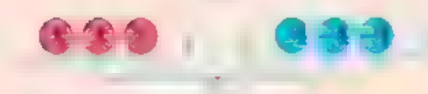
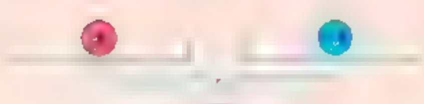
Pictorial representation	Algebraic representation
Since RHS = $2x - 8 = 2x + (-8)$, we can represent the equation as follows: 	$5x + 4 = 2x - 8$
Take away 2 discs from each of the two pans: 	Subtract $2x$ from both sides of the equation: $5x + 4 - 2x = 2x - 8 - 2x$
	Simplify both sides of the equation: $3x + 4 = -8$
Since we cannot take away from the RHS, we add to each of the two pans (because subtracting is the same as adding): zero pair 	Subtract 4 from both sides of the equation: $3x + 4 - 4 = -8 - 4$
	Simplify both sides of the equation: $3x = -12$
	Divide both sides of the equation by 3: $\frac{3x}{3} = \frac{-12}{3}$
	Simplify both sides of the equation: $x = -4$

Table 7.3

4. Solve each of the following equations using the idea of the balancing scale.
(a) $5x + 3 = 3x - 7$ (b) $4x - 2 = x + 7$ (c) $3x - 2 = -x + 14$ (d) $-2x - 5 = 5x - 12$
For part (d), you cannot take away 5 discs from the RHS. What can you add to both sides of the equation instead?

From the Investigation on pages 162 and 163, we learnt that in order to find the value of the unknown x that *satisfies* the equation, we need to *manipulate* the equation in order to *isolate* x on one side of the equation.

We also observe that as we manipulate the linear equation from one equation to another, all the equations have the same solution. For example, all the following equations in Part 3 of the Investigation have the same solution $x = -4$:

- $5x + 4 = 2x - 8$
- $3x + 4 = -8$
- $3x = -12$
- $x = -4$

Two equations are equivalent if they have the same solution, e.g. $5x + 4 = 2x - 8$ and $3x + 4 = -8$ have the same solution $x = 4$

All the above equations are **equivalent**. The basic steps to solve a linear equation involve the conversion of the equation to another equivalent equation, until the unknown is isolated on one side of the equation.

Solving linear equations

Solve each of the following equations.

(a) $2x + 3 = 9$ **(b)** $2(y - 4) = 5y - 3$

(a)

$$2x + 3 = 9$$

$$2x + 3 - 3 = 9 - 3$$

$$2x = 6$$

$$\frac{2x}{2} = \frac{6}{2} \quad \text{divide both sides by 2}$$

$$x = 3$$

Problem-solving Tip

It is a good practice to *check* your solution by substituting the value of the unknown that you have found into the original equation to see if it satisfies the equation, e.g. in **(a)**, LHS = $2(3) + 3 = 6 + 3 = 9 =$ RHS.

(b) **Method 1:**

$$2(y - 4) = 5y - 3$$

$$2y - 8 = 5y - 3 \quad \text{Distributive Law}$$

$$2y - 5y - 8 = 5y - 5y - 3 \quad \text{subtract } 5y \text{ from both sides}$$

$$-3y - 8 = -3$$

$$-3y - 8 + 8 = -3 + 8 \quad \text{add 8 to both sides}$$

$$-3y = 5$$

$$\frac{-3y}{-3} = \frac{5}{-3} \quad \text{divide both sides by } (-3)$$

$$y = -\frac{5}{3}$$

$$= -1\frac{2}{3}$$

Method 2:

$$\begin{aligned}
2(y-4) &= 5y-3 \\
2y-8 &= 5y-3 && \text{Distributive Law} \\
5y-3 &= 2y-8 && \text{change sides} \\
5y-3-2y &= 2y-8-2y && \text{subtract } 2y \text{ from both sides} \\
3y-3 &= -8 \\
3y-3+3 &= -8+3 && \text{add 3 to both sides} \\
3y &= -5 \\
\frac{3y}{3} &= \frac{-5}{3} && \text{divide both sides by 3} \\
y &= -\frac{5}{3} \\
&= -1\frac{2}{3}
\end{aligned}$$

Reflection

(b) Which method do you prefer? Why?

Practise Now

Solve each of the following equations.

(a) $3x + 2 = 8$ (b) $5x - 3 = 3x - 12$

(c) $2(2y + 1) = 9y + 4$ (d) $2(z - 1) + 3(z - 1) = 4 + 2z$

Solve each of the following equations.

(a) $3x + 4 = 2x + 1$ (b) $3x + 4 = 3x + 1$ (c) $3(x + 2) = 3x + 6$

What do you notice? Can you explain why?

From the above Thinking Time, we observe that there are at least three types of equations:

(a) Conditional equation: An equation that is true only for *some* values of the variable, e.g. $3x + 4 = 2x + 1$.

(b) Equation that is a contradiction: An equation that is *not* true for any value of the variable, e.g. $3x + 4 = 3x + 1$.

(c) Identity: An equation that is true for *all* values of the variable, e.g. $3(x + 2) = 3x + 6$.

In Section 7.3, we will learn another type of equation called a formula.

$2x + 3 = 9$, $2x = 6$, $x = 3$ and $10x - 4 = 5x + 11$ are known as **equivalent equations** because they have the same solution, i.e. $x = 3$. State three equivalent equations that have the same solution $x = -1$.

1. What is the difference between an expression and an equation? Can I solve an expression? Explain.
2. What are some rules to apply when solving a linear equation?
3. Can all linear equations be solved? Why or why not?

7.2 Linear equations with fractional coefficients and fractional equations

In Chapter 6, we have learnt about linear expressions with fractional coefficients. After learning how to solve linear equations with integer coefficients earlier in this chapter, we will now solve linear equations with fractional coefficients (see Worked Example 2), and another type of equations called fractional equations (see Worked Example 3).

2

Solving linear equations with fractional coefficients

Solve each of the following equations.

(a) $\frac{5}{2}x+3\frac{1}{2}=\frac{2}{3}x+5$ **(b)** $\frac{z+2}{3}=\frac{3z+2}{5}$

Solution

(a) Method 1:

$$\frac{5}{2}x+3\frac{1}{2}=\frac{2}{3}x+5$$

$$\frac{5}{2}x-\frac{2}{3}x+3\frac{1}{2}=\frac{2}{3}x-\frac{2}{3}x+5 \quad \text{subtract } \frac{2}{3}x \text{ from both sides}$$

$$\frac{15}{6}x-\frac{4}{6}x+3\frac{1}{2}=5$$

$$\frac{11}{6}x+3\frac{1}{2}=5$$

$$\frac{11}{6}x+3\frac{1}{2}-3\frac{1}{2}=5-3\frac{1}{2} \quad \text{subtract } 3\frac{1}{2} \text{ from both sides}$$

$$\frac{11}{6}x=1\frac{1}{2}$$

$$\frac{11}{6}x\times\frac{6}{11}=1\frac{1}{2}\times\frac{6}{11} \quad \text{multiply both sides by } \frac{6}{11}$$

$$x=\frac{3}{2}\times\frac{6}{11}$$

$$=\frac{9}{11}$$

Method 2:

$$\frac{5}{2}x+3\frac{1}{2} = \frac{2}{3}x+5$$
$$\frac{5}{2}x+\frac{7}{2} = \frac{2}{3}x+5$$
$$\left(\frac{5}{2}x+\frac{7}{2}\right)\times 6 = \left(\frac{2}{3}x+5\right)\times 6$$
$$15x + 21 = 4x + 30$$
$$15x + 21 - 4x = 4x + 30 - 4x \quad \text{subtract } 4x \text{ from both sides}$$
$$11x + 21 = 30$$
$$11x + 21 - 21 = 30 - 21 \quad \text{subtract 21 from both sides}$$
$$11x = 9$$
$$x = \frac{9}{11}$$

Reflection

(a) Which method do you prefer? Why?

(b) $$\frac{z+2}{3} = \frac{3z+2}{5}$$
$$\frac{z+2}{3}\times 15 = \frac{3z+2}{5}\times 15 \quad \text{multiply both sides by LCM of denominators}$$
$$5(z+2) = 3(3z+2)$$
$$5z + 10 = 9z + 6 \quad \text{Distributive Law}$$
$$5z - 9z = 6 - 10 \quad \text{subtract } 9z \text{ and 10 from both sides}$$
$$-4z = -4$$
$$\frac{-4z}{-4} = \frac{-4}{-4} \quad \text{divide both sides by } (-4)$$
$$z = 1$$

Problem-solving Tip

(b) At this stage, you can subtract $9z$ and 10 from both sides of the equation at the same time to shorten the working.

Exercise 7A

1. Solve each of the following equations.

   (a) $\frac{5}{7}y+2 = \frac{1}{2}y+3\frac{1}{4}$ (b) $\frac{3z-1}{2} = \frac{z-4}{3}$

2. Solve each of the following equations.

   (a) $x + 0.7 = 1.9$ (b) $2y - 1.3 = 2.8$

Solving fractional equations

Solve each of the following equations.

(a) $\dfrac{9}{2x-5}=3$ (b) $\dfrac{y+4}{2y-3}=\dfrac{2}{5}$

> **Attention**
> These are *fractional equations*, which are different from linear equations with fractional coefficients. To solve them, we convert them into linear equations with integer coefficients.

Solution

(a)

$$\frac{9}{2x-5}=3$$

$$\frac{9}{2x-5}\times(2x-5)=3(2x-5) \quad \text{multiply both sides by } (2x-5)$$

$$9=6x-15 \quad \text{Distributive Law}$$

$$6x-15=9 \quad \text{change sides}$$

$$6x=24 \quad \text{add 15 to both sides and simplify}$$

$$x=4 \quad \text{divide both sides by 6 and simplify}$$

(b)

$$\frac{y+4}{2y-3}=\frac{2}{5}$$

$$\frac{y+4}{2y-3}\times 5(2y-3)=\frac{2}{5}\times 5(2y-3)$$

$$5(y+4)=2(2y-3)$$

$$5y+20=4y-6 \quad \text{Distributive Law}$$

$$5y-4y=-6-20 \quad \text{subtract } 4y \text{ and 20 from both sides}$$

$$y=-26$$

Practice Now 3

Solve each of the following equations.

(a) $\dfrac{8}{2x-3}=4$ (b) $\dfrac{y-3}{y+4}=\dfrac{3}{2}$

1. What have I learnt about solving linear equations with integer coefficients in Section 7.1 that helped me to solve linear equations with fractional coefficients, and fractional equations?
2. When I encounter a linear equation with fractional coefficients, or a fractional equation, what is the first step that I should take to solve the equation?

Exercise 7A

1. Solve each of the following equations.

(a) $x + 8 = 15$ (b) $x + 9 = -5$
(c) $x - 5 = 17$ (d) $y - 7 = -3$
(e) $4x = -28$ (f) $-24x = -144$
(g) $3x - 4 = 11$ (h) $9x + 4 = 31$
(i) $12 - 7x = 6$ (j) $3 - 7y = -12$

2. Solve each of the following equations.

(a) $3x - 7 = 4 - 8x$ (b) $4x - 10 = 5x + 7$
(c) $30 + 7y = -2y - 6$ (d) $2y - 7 = 7y - 23$

3. Solve each of the following equations.

(a) $2(x + 3) = 8$ (b) $5(x - 7) = -15$
(c) $7(-2x + 4) = -4x$ (d) $3(2y + 3) = 4y + 3$
(e) $2(y + 4) = 3(y + 2)$ (f) $5(5y - 6) = 4(y - 7)$
(g) $5(b + 6) = 2(3b - 4)$ (h) $3(2c + 5) = 4(c - 3)$
(i) $9(2d + 7) = 11(d + 14)$ (j) $28(f - 1) = 5(7f - 3)$

4. Solve each of the following equations.

(a) $7y - 2\frac{3}{4} = \frac{1}{2}$ (b) $1\frac{1}{2} - 2y = \frac{1}{4}$
(c) $\frac{1}{3}x = 7$ (d) $\frac{3}{4}x = -6$
(e) $\frac{1}{3}x + 3 = 4$ (f) $\frac{y}{4} - 8 = -2$
(g) $3 - \frac{1}{4}y = 2$ (h) $15 - \frac{2}{5}y = 11$

5. Solve each of the following equations.

(a) $y - 2.4 = 3.6$ (b) $y + 0.4 = 1.6$
(c) $-3y - 7.8 = -9.6$ (d) $4y - 1.9 = 6.3$
(e) $-2.7 + a = -6.4$ (f) $2(2x - 2.2) = 4.6$
(g) $4(3y + 4.1) = 7.6$ (h) $3(2 - 0.4x) = 18$

6. Solve each of the following equations.

(a) $x = 12 - \frac{1}{3}x$ (b) $\frac{3}{5}x = \frac{1}{2}x + \frac{1}{2}$
(c) $\frac{y}{2} - \frac{1}{5} = 2 - \frac{y}{3}$ (d) $\frac{2}{3}y - \frac{3}{4} = 2y + \frac{5}{8}$

7. Solve each of the following equations.

(a) $\frac{2}{x} = \frac{4}{5}$ (b) $\frac{12}{y-1} = \frac{2}{3}$

8. Solve each of the following equations.

(a) $-3(2 - x) = 6x$
(b) $5 - 3x = -6(x + 2)$
(c) $-3(9y + 2) = 2(-4y - 7)$
(d) $-3(4y - 5) = -7(-5 - 2y)$
(e) $3(5 - h) - 2(h - 2) = -1$

9. Solve each of the following equations.

(a) $10x - \frac{5x+4}{3} = 7$
(b) $\frac{4x}{3} - \frac{x-1}{2} = 1\frac{1}{4}$
(c) $\frac{x-1}{3} - \frac{x+3}{4} = -1$
(d) $1 - \frac{y+5}{3} = \frac{3(y-1)}{4}$
(e) $\frac{6(y-2)}{7} - 12 = \frac{2(y-7)}{3}$
(f) $\frac{7-2y}{2} - \frac{2}{5}(2-y) = 1\frac{1}{4}$

10. Solve each of the following equations.

(a) $\frac{5x+1}{3} = 7$ (b) $\frac{2x-3}{4} = \frac{x-3}{3}$
(c) $\frac{3x-1}{5} = \frac{x-1}{3}$ (d) $\frac{1}{4}(5y+4) = \frac{1}{3}(2y-1)$
(e) $\frac{2y-1}{5} - \frac{y+3}{7} = 0$ (f) $\frac{2y+3}{4} + \frac{y-5}{6} = 0$

11. By showing your working clearly, verify if $x = \frac{19}{20}$ is the solution of the equation $2x - \frac{3}{4} = \frac{1}{3}x + \frac{5}{6}$.

12. Solve each of the following equations.

(a) $\frac{12}{x+3} = 2$ (b) $\frac{11}{2x-1} = 4$
(c) $\frac{32}{2x-5} - 3 = \frac{1}{4}$ (d) $\frac{1}{2} = \frac{1}{x+2} - 1$
(e) $\frac{y+5}{y-6} = \frac{5}{4}$ (f) $\frac{2y+1}{3y-5} = \frac{4}{7}$
(g) $\frac{2}{y-2} = \frac{3}{y+6}$ (h) $\frac{2}{7y-3} = \frac{3}{9y-5}$

Exercise 7A

If $4x + y = 3x + 5y$, find the value of $\frac{3x}{16y}$.

If $\frac{3x-5y}{7x-4y} = \frac{3}{4}$, find the value of $\frac{x}{y}$.

7.3 Applications of linear equations in real-world contexts

Algebra is useful to solve word problems that we are unable to easily draw models for, e.g. the word problem in the Introductory Problem.
In addition, not all word problems can be solved by drawing models. For example, try solving the following problem.

> A man is 5 times as old as his son. Four years ago, the product of their ages was 52. Find their present ages.

Drawing models cannot solve the above problem, which we will learn how to solve using algebra in Book 2.
In this section, we will learn how to solve word problems involving linear equations, using algebra. But first, let us see how the use of models can be linked to algebra.

A. Linking models to algebra

Consider the following problem:

> A man is 3 times as old as his son. In 10 years' time, the sum of their ages will be 76.
> How old was the man when his son was born?

In Table 7.4 on the next page, the column on the left shows how the problem is solved by drawing a model.
Replacing each of the boxes with the unknown x, we obtain $4x = 76 - 20$.
The column on the right shows how the problem is solved using algebra. Do you notice the similarities between each corresponding step of the two methods?
The steps for both methods are similar, apart from the first step of forming the equation in the right column, which is simplified to . Thus, we see the link between the two methods.

Model	Algebra
Son $\boxed{x}$	Let the age of the son be x years.
Man $\boxed{x}\boxed{x}\boxed{x}$	Then the man is $3x$ years old.
In 10 years' time,	In 10 years' time,
Son $\boxed{x}\boxed{10}$	the son will be $(x + 10)$ years old
Man $\boxed{x}\boxed{x}\boxed{x}\boxed{10}$ } 76	and the man will be $(3x + 10)$ years old.
	$\therefore (x + 10) + (3x + 10) = 76$
	$4x + 20 = 76$
$4x = 76 - 20$	$4x = 76 - 20$
$= 56$	$= 56$
$x = \frac{56}{4}$	$x = \frac{56}{4}$
$= 14$	$= 14$
When his son was born, the man was $2 \times 14 = 28$ years old.	When his son was born, the man was $2x = 2 \times 14 = 28$ years old.

Table 7.4

B. Solving problems using algebra

Let us now use algebra to solve word problems, including those in real-world contexts.

Worked Example 4

Solving problem involving equation with integer coefficients

The sum of three consecutive even numbers is 60. Find the numbers.

Solution

Let the smallest even number be n.

Then the other two consecutive even numbers will be $n + 2$ and $n + 4$.

$\therefore$ sum of the three consecutive even numbers $= n + (n + 2) + (n + 4) = 60$

$$
\begin{aligned}
3n + 6 &= 60 \\
3n &= 60 - 6 \\
&= 54 \\
n &= \frac{54}{3} \\
&= 18
\end{aligned}
$$

$\therefore$ the three consecutive even numbers are 18, 20 and 22.

Exercise 7B

1. The sum of two numbers, one of which is 5 times as large as the other, is 24. Find the two numbers.
2. In a science test, Shaha's score is 15 marks more than Cheryl's. If Shaha obtains twice as many marks as Cheryl, find the number of marks Cheryl obtains.

Introductory Problem Revisited

After learning how to solve word problems using algebra, how do you use the same method to solve the ? Is it much easier? Discuss your solution with your classmates.

Worked Example 5

Solving problem involving equation with fractional coefficients

Ali walked at an average speed of 3 km/h for 45 minutes before running for half an hour at a certain average speed. If he travelled a total distance of 6 km, calculate his average running speed.

Total distance walked = average walking speed × walking time

$$= 3 \times \frac{45}{60}$$

$$= \frac{9}{4} \text{ km}$$

Let Ali's average running speed be x km/h.

Total distance ran = average running speed × running time

$$= x \times \frac{1}{2}$$

$$= \frac{x}{2} \text{ km}$$

$\therefore$ total distance travelled $= \frac{9}{4} + \frac{x}{2}$

$$\therefore \frac{9}{4} + \frac{x}{2} = 6$$

$$9 + 2x = 24 \quad \text{multiply both sides by LCM of denominators}$$

$$2x = 15$$

$$x = 7.5$$

$\therefore$ Ali's average running speed = 7.5 km/h

Attention

Since the average running speed is x km/h, then x is a number without any unit.

Problem-solving Tip

Since the walking speed is in km/h, we need to convert the walking time of 45 min to $\frac{45}{60}$ h before we can calculate the walking distance in km.

The sum of one-fifth of a number and $3\frac{7}{10}$ is 7. Find the number.

1. How is using algebra to solve word problems similar to/different from using models?
2. What is/are the challenging step(s) involved in using algebra to solve word problems?

Exercise 7B

Use algebra to solve the following questions.

1. When loaded with bricks, a lorry has a mass of 11 600 kg. If the mass of the bricks is three times that of the empty lorry, find the mass of the bricks.

2. The sum of 4 consecutive odd numbers is 56. Find the greatest of the 4 numbers.

3. David is 4 years older than Sara and Yasir is 2 years younger than Sara. If the sum of their ages is 47, find their respective ages.

4. If a number is tripled, it gives the same result as when 28 is added to it. Find the number.

5. A travel agency is planning a holiday for a group of people. The agency receives quotations from two coach companies, Maya Express and Great Holidays. Maya Express charges $15 for each person while Great Holidays charges $12 per person and a separate fee of $84. If the total amount charged by each company is the same, find the number of people going on the holiday.

6. The sum of two numbers, one of which is two-thirds of the other, is 45. Find the smaller number.

7. When a number, x, is multiplied by 4, then subtracted from 68, the result obtained is the same as three times the sum of x and 4. Find x.

8. In a school, the number of boys who play soccer is three times of those who play badminton. If 12 boys who play soccer play badminton instead, the number of boys who play each of these sports would be the same. Find the number of boys who play badminton.

9. A man is six times as old as his son. Twenty years later, the man will be twice as old as his son. Find the age of the man when his son was born.

10. A chocolate cake costs $2 more than a butter cake. The cost of 6 chocolate cakes and 5 butter cakes is $130.80. Find the cost of a chocolate cake.

11. Albert has 12 more 10-cent coins than 20-cent coins. The total value of all his coins is $5.40. Find the total number of coins he has.

12. Li Ting cycles the first 350 km of a 470-km journey at a certain average speed and the remaining distance at an average speed that is 15 km/h less than that for the first part of the journey. If the time taken for her to travel each part of her journey is the same, find the average speed for the second part of her journey.

13. The sum of half of a number and 49 is $2\frac{1}{4}$ times of the number. Find the number.

14. The numerator of a fraction is 5 less than its denominator. If 1 is added to the numerator and to the denominator, the new fraction is $\frac{2}{3}$. Find the fraction.

15. A two-digit positive number is such that the ones digit is 2.5 times as much as the tens digit. If the difference between the number and the number obtained when the digits are reversed is 27, find the number.

7.4 Mathematical formulae

In primary school, we have learnt that the area A of a rectangle is the product of its length l and its breadth b: $A = l \times b$ or $A = lb$. This is called a **formula**. A formula (plural: formulae) is an *equation* that expresses the *relationship* between certain quantities or variables. In this case, the variables are A, l and b.

Formulae are useful to help us find the unknown value of a variable if we know the values of all the other variables. Name a few other mathematical formulae that you have learnt.

A. Evaluating formulae

Worked Example 6

Finding unknown in formula

The formula for finding the volume V of a cuboid is given by $V = lbh$, where l, b and h represent the length, the breadth and the height of the cuboid respectively.

(a) If $l = 5$ cm, $b = 2$ cm and $h = 3$ cm, calculate the volume of the cuboid.

(b) If $V = 240$ cm^3, $b = 6$ cm and $h = 5$ cm, calculate the length of the cuboid.

Solution

(a) $V = lbh$

When $l = 5, b = 2, h = 3$, $V = 5 \times 2 \times 3$
$= 30$ cm^3

$\therefore$ volume of cuboid $= 30$ cm^3

(b) $V = lbh$

When $V = 240, b = 6, h = 5$, $l \times 6 \times 5 = 240$

$$30l = 240$$
$$l = \frac{240}{30}$$
$$= 8 \text{ cm}$$

$\therefore$ length of cuboid $= 8$ cm

Exercise 7C

1. Newton's second law states that the net force F acting on a body is given by $F = ma$, where m is the mass and a is the acceleration of the body. The units for F, m and a are the Newton (N), the kilogram (kg) and metre per square second (m/s^2) respectively.
   (a) If $m = 1000$ kg and $a = 0.05$ m/s^2, find the net force acting on the body.
   (b) If $F = 100$ N and $a = 0.1$ m/s^2, find the mass of the body.
2. If $y + b = \frac{ay + c}{b}$, calculate the value of c when $y = 12$, $b = 3$ and $a = 14$.

B. Constructing formulae

To construct a formula, choose letters to represent the quantities and express their relationship in an equation. Usually, the first letter of the quantity is used, e.g. we use the letter S to represent the sum of any three consecutive odd numbers.

Constructing formula

(i) Find a formula for the sum S of any three consecutive odd numbers.

(ii) Hence, find the value of S if the greatest odd number is -101.

(i) Let the smallest odd number be n.
Then the other two consecutive odd numbers will be $n + 2$ and $n + 4$.
$\therefore$ sum of the three consecutive odd numbers,

$$S = n + (n + 2) + (n + 4)$$
$$= 3n + 6, \text{ where } n \text{ is odd}$$

(ii) If the greatest odd number is -101, then

$$n + 4 = -101$$
$$n = -101 - 4$$
$$= -105$$

$$\therefore S = 3n + 6$$
$$= 3 \times (-105) + 6$$
$$= -309$$

Attention

$S = 3n + 6$ is the same formula as that in Worked Example 4 for the sum of three consecutive even numbers. The difference is that n is odd in this Worked Example while in Worked Example 4, n is even.

(i) Find a formula for the sum S of any four consecutive even numbers.

(ii) Hence, calculate the value of S when the smallest even number is 14.

1. How is finding an unknown in a formula similar to/different from solving an equation?
2. What have I learnt in this section or chapter that I am still unclear of?

Exercise 7C

1. If $y = \frac{3}{5}x + 26$, find the value of y when $x = 12$.

2. If $S = 4\pi r^2$, find
   (i) the value of S when $r = 10\frac{1}{2}$,
   (ii) the positive value of r when $S = 616$.
   (Take π to be $\frac{22}{7}$.)

3. If $a = \frac{y^2 - xz}{5}$, find the value of a when $x = 2$, $y = -1$, and $z = -3$.

4. If $k = \frac{p + 2q}{3}$, find the value of p when $k = 7$ and $q = 9$.

5. Find a formula for each of the following.
   (a) Product P of three numbers x, y and z
   (b) Sum S of the square of p and the cube of q
   (c) Average A of four numbers m, n, p and q
   (d) Time T, in minutes, for a train journey of a hours b minutes

6. If $U = \pi(r + h)$, find the value of r when $U = 16\frac{1}{2}$ and $h = 2$. (Take π to be $\frac{22}{7}$.)

7. If $v^2 = u^2 + 2gs$, find the value of s when $v = 25$, $u = 12$ and $g = 10$.

8. If $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} + \frac{1}{d}$, find the value of c when $a = \frac{1}{2}$, $b = \frac{1}{4}$ and $d = -\frac{1}{5}$.

9. If $c = \frac{a}{b} - \frac{d - e}{f - d}$, find the value of f when $a = 3$, $b = 4$, $c = -6$, $d = -5$ and $e = 2$.

10. If $\frac{m(nx^2 - y)}{z} = 5n$, find the value of n when $m = 6$, $x = -2$, $y = -3$ and $z = -5$.

11. (i) Find a formula for the sum S of any four consecutive odd numbers.
    (ii) Hence, find the value of S when the greatest odd number is -17.

12. Joyce has four siblings, with an age gap of two years between each child.
    (i) Find a formula for the sum of the total ages of Joyce and her siblings.
    (ii) Hence, if the youngest child is now 4 years old, find the sum of their total ages at the end of the following year.

13. (i) Find a formula for the total cost $\$T$ of c pens at $\$d$ each and e pencils at f cents each.
    (ii) If $e = \frac{-145c}{4 - c}$ and $d = \frac{f + 5}{50}$, where $e = 150$ and $d = 3$, find the value of T.

14. In the United States of America (USA), a different unit is used to measure temperature. It is called the degree Fahrenheit (°F). The formula for the conversion of x °F to y degree Celsius (°C) is

$$y = (x - 32) \times \frac{5}{9}.$$

    (i) The highest temperature in the USA, recorded in Death Valley in California, is 134 °F. What is this temperature in °C?
    (ii) During winter in the USA, it is very common for the temperature to fall below 0 °C. Is it more or less common for the temperature to fall below 0 °F?
    (iii) The lowest temperature in the USA, recorded in Prospect Creek in Alaska, is -62.1 °C. What is this temperature in °F?

As we continue on this exciting journey of learning algebra, we may encounter some difficulties in communicating mathematical ideas using the language of algebra. For example, when does the algebraic notation, e.g. x, refer to a specific unknown number or a variable? However, the power of algebraic **notation** still outweighs the difficulties. For instance, the same description in the Chapter Opener can be expressed using algebra as follows:

Let x be the unknown quantity. Then we have:

$$\begin{aligned} \frac{3}{2}x+4 &= 10 \\ \frac{3}{2}x &= 6 \\ x &= 6\times\frac{2}{3} \\ &= 4 \end{aligned}$$

The key to solving linear equations is the idea of **equivalence**, which helps us to see why and how algebraic equations can be manipulated to isolate the unknown. This is a basic but important idea that empowers us to work with more complicated algebraic expressions to **model** real-world situations.

Summary

1. **Solving of equations**

   To solve an equation in x means to find the value(s) of x for which the equation is true.
   This usually involves *isolating* x on one side of the equation.
   For linear equations, we can isolate x by adding, subtracting, multiplying or dividing by the same number on both sides of the equation.
   For example, to solve $x - 2 = 7$, we add 2 to both sides of the equation:

   $x - 2 = 7$
   $x \quad 2 + 2 = 7 + 2$
   $x = 9$

   - Give an example of a linear equation where you have to subtract the same number from both sides of the equation, and solve it.
   - Give an example of a linear equation where you have to multiply or divide both sides of the equation by the same number, and solve it.
   - Give an example of a linear equation where you have to do at least two of the four operations (of addition, subtraction, multiplication and division), and solve it.

2. **Solving of linear equations with fractional coefficients, and fractional equations**

   One method to solve a linear equation with fractional coefficients, or a fractional equation, is to convert it into a linear equation with integer coefficients by multiplying both sides of the equation by the LCM of the denominators of the fractions.

   An example of a linear equation with fractional coefficients is $\frac{5}{2}x + 3\frac{1}{2} = \frac{2}{3}x + 5$.

   An example of a fractional equation is $\frac{9}{2x-5} = 3$.

   - Give an example of a linear equation with fractional coefficients, and solve it.
   - Give an example of a fractional equation, and solve it.

3. **Mathematical formulae**

   A formula is an equation that expresses the relationship between certain quantities or variables in algebraic terms, e.g. $A = l \times b$. We can calculate an unknown value when given the values of other variables.

   - Give an example of a formula and explain why you classify it as such.

CHAPTER 8

Percentage

Augustus Caesar, the first Emperor of the Roman Empire, levied a tax of $\frac{1}{100}$ (or 1%) on goods sold at auctions. This tax was known as *centesima rerum venalium*, which literally translates to 'a hundredth of the value of everything sold'. During the Middle Ages of Europe (5th to 15th century), computations with a denomination of 100 became ubiquitous. By the 17th century, interest rates were quoted in terms of hundredths. The word 'percent' has its roots in the Latin phrase *per centum*, which means 'by the hundred'.

In our modern society, percentages are widely used in businesses and are also frequently employed to convey statistical information in reports. What are some other uses of percentage?

Learning Outcomes

What will we learn in this chapter?

- What percentage and percentage change are
- How to use percentages greater than 100%
- How to compare two quantities by percentage

Introductory Problem

Fig. 8.1 shows the enrolment at various education stages in the years 2020 to 2021 in the provinces of Punjab and Sindh.

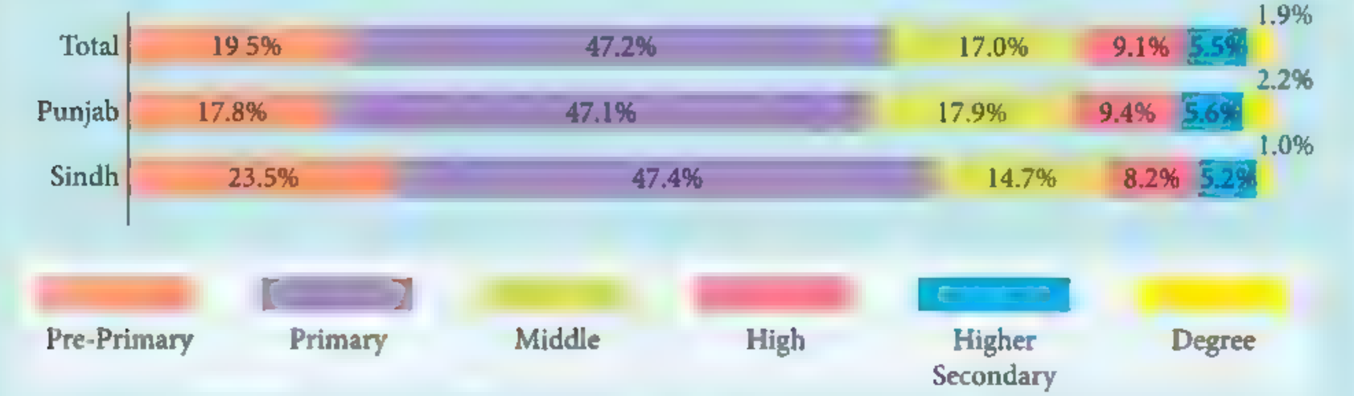


Fig. 8.1

Source: Pakistan [illegible] NEMIS PIE [illegible]

1. Can you tell how many students from the province of Punjab are enrolled in middle school or below? If yes, explain how. If not, explain why not.
2. Can you tell if the total enrolment in pre-primary, primary and middle schools in Sindh is higher than that in Punjab? If yes, explain how. If not, explain why not.
3. Since 9.4% and 8.2% of the students from Sindh and Punjab are respectively enrolled in high schools, why is the total percentage in the first row of Fig. 8.1 only 9.1%?

In this chapter, we are going to learn more about how percentages are calculated in various contexts, so that we can correctly interpret the information presented.

8.1 Percentage

Percentages are used to convey information in everyday life. We often see advertisements or newspaper reports with headlines such as 'Warehouse Sale at 50% off' and 'Gross Domestic Product is up by 0.5%'.

50% and 0.5% are examples of percentages.

The symbol, %, is used to represent '**percent**'. Thus 50% is read as '50 percent'

Information

The word '**percent**' originated from the Latin phrase '*per centum*', which means 'per hundred' or 'out of every hundred'

Identifying percentages used in daily life

Read the following advertisement on grapefruits.

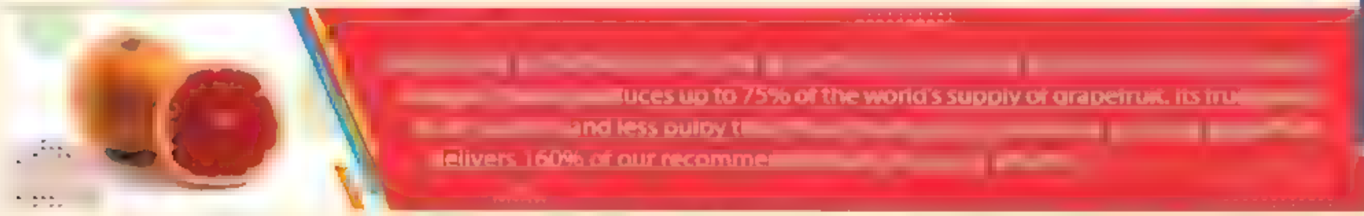


Fig. 8.2

(i) Discuss with your classmates how to interpret the various percentages found in the advertisement.

(ii) Does 'up to 75% of the world's supply of grapefruit' mean that Florida produces 75% of the world's supply of grapefruit?

(iii) Can a percentage be greater than 100%? Consider the phrase 'one grapefruit delivers 160% of our recommended daily intake of vitamin C' in the advertisement. What does it mean?

A. Percentage, fraction and decimal

In primary school, we have learnt that x percent is defined as x parts per hundred, i.e.

$$x\% = \frac{x}{100}.$$

Attention

$100\% = \frac{100}{100} = 1$

Therefore, a percentage can be expressed as a fraction, or a decimal, and vice versa.
For example,

Percentage to fraction	Fraction to percentage	Percentage to decimal	Decimal to percentage
$72\% = \frac{72}{100}$ $= \frac{18}{25}$	$\frac{18}{25} = \frac{18}{25} \times 100\%$ $= 72\%$	$72\% = \frac{72}{100}$ $= 0.72$	$0.72 = 0.72 \times 100\%$ $= 72\%$

1. Express each of the following percentages as a fraction and as a decimal.
   (a) 76% (b) 9%

2. Express each of the following fractions or decimals as a percentage.
   (a) $\frac{13}{20}$ (b) 0.25

Worked Example 1

Conversion involving non-integer percentages

(a) Express each of the following percentages as a fraction and as a decimal.

(i) $4\frac{2}{3}\%$ **(ii)** 22.5%

(b) Express each of the following as a percentage.

(i) $\frac{5}{6}$ **(ii)** 0.769

Solution

(a) (i)

$$4\frac{2}{3}\% = 4\frac{2}{3} \div 100$$
$$= \frac{14}{3} \times \frac{1}{100}$$
$$= \frac{7}{150}$$

$$4\frac{2}{3}\% = 4\frac{2}{3} \div 100$$
$$= 4.67 \div 100 \text{ (to 3 s.f.)}$$
$$= 0.0467$$

(ii)

$$22.5\% = \frac{22.5}{100}$$
$$= \frac{225}{1000} \quad \text{HCF of 225 and 1000 is 25}$$
$$= \frac{9}{40}$$

$$22.5\% = \frac{22.5}{100}$$
$$= 0.225$$

(b) (i)

$$\frac{5}{6} = \frac{5}{6} \times 100\% \quad \text{since } 100\% = 1$$
$$= \frac{250}{3}\%$$
$$= 83\frac{1}{3}\%$$

(ii)

$$0.769 = 0.769 \times 100\%$$
$$= 76.9\%$$

Equivalence

For **(a)(i)**, we could also have converted $\frac{7}{150}$ to 0.0467 directly. Since $4\frac{2}{3}\% = \frac{7}{150} = 0.0467$, we say that these different forms have equivalent values.

Attention

(a) (i) Leave non-exact answers correct to 3 significant figures, unless the question states otherwise.

Attention

(b) (i) You can also leave your answer as 83.3% (to 3 s.f.).

Practise Now 1

Exercise 8A

1. Express each of the following percentages as a fraction and as a decimal.

(a) $93\frac{5}{6}\%$ **(b)** 14.7%

2. Express each of the following as a percentage.

(a) $\frac{4}{7}$ **(b)** 0.5432

B. Percentages greater than 100% and percentages less than 1%

A percentage can be greater than 100%.

For example, from the Class Discussion on page 181, 160%, read as 'one hundred and sixty percent', means $\frac{160}{100}$ or 1.6.

A percentage can also be less than 1%, e.g. 0.5%.

Example 2

Expressing percentages greater than 100% or less than 1% as decimals

Express each of the following percentages as a decimal.

(a) 238.7% (b) 0.5%

Solution

(a) $238.7\% = \frac{238.7}{100} = 2.387$

(b) $0.5\% = \frac{0.5}{100} = 0.005$

Attention

(a) 238.7% is different from 238.7. Percentages greater than 100% are equal to decimals greater than 1, or mixed numbers.

(b) 0.5% is different from 0.5. Percentages smaller than 1% are equal to decimals less than 0.01, or fractions less than $\frac{1}{100}$.

Exercise 8A

1. Express each of the following percentages as a fraction/mixed number and as a decimal.

   (a) 318% (b) $407\frac{1}{4}\%$

   (c) 0.2% (d) 0.066%

2. Express each of the following numbers as a percentage.

   (a) $49\frac{1}{3}$ (b) 5.468

   (c) 0.0016 (d) $\frac{17}{2000}$

We can change any fraction or decimal into a percentage by multiplying it by 100%:

$$\frac{p}{q} = \frac{p}{q} \times 100\%.$$

But we cannot multiply it by 100:

$$\frac{p}{q} \neq \frac{p}{q} \times 100.$$

Explain why this is so.

C. Percentage of a quantity

In primary school, we have learnt how to express a part of a whole as a percentage and vice versa. Let us look at some examples of how such percentages are used in real life.

Interpreting percentages used in real life

Part 1

Table 8.1 shows the shooting accuracy of the top 5 soccer players during the FIFA World Cup 2014 after the teams had reached the quarter-final stage.

Player	Team	Shooting accuracy (5+ shots)
Giovani dos Santos	Mexico	100.0%
Arjen Robben	Netherlands	88.9%
James Rodriguez	Colombia	87.5%
Ahmed Musa	Nigeria	83.4%
Neymar	Brazil	81.9%

Table 8.1

Source: Mail Online, 2 July 2014

(a) Let x be the total number of shots attempted by Neymar. How many goals did he score? Give your answer as a decimal in terms of x.

(b) The shooting accuracy of Giovani dos Santos was 100.0%, which is greater than that of Neymar. Can we conclude that Giovani dos Santos had scored more goals than Neymar in the FIFA World Cup 2014? Explain your answer.

Part 2

Table 8.2 shows the results of the 2022 parliamentary election of a particular country.

Constituency (Number of electors)	Name of party	Total number of valid votes	Total valid votes (%)
ABC New Town (179 071)	P	112,677	69.33
	Q	49,851	30.67

Table 8.2

After reading the above, Joyce was very puzzled and asked, "ABC New Town had 179 071 electors in the year 2022, and Party P won by obtaining 69.33% of the votes. But 69.33% of 179 071 is not equal to 112 677. Is there an error?"

(a) Explain to Joyce how the percentage 69.33% was derived.

(b) What percentage of the electors had made valid votes?

From the above Class Discussion, we see that a percentage is a proportion in relation to a whole. Thus, x% of quantity A may not be equal to x% of quantity B.

In general, a percentage helps us to visualise how a given quantity is divided proportionally.

Attention

The word 'proportion' used in relation to percentage refers to a part-whole ratio. Recall that we learnt about ratio in primary school.

Finding percentage of quantity

A class has 40 students of which 15 are boys and the rest are girls.

(a) If 25% of the class wear spectacles, find the number of students who do not wear spectacles.

(b) If 60% of the girls have long hair, find the number of girls with long hair.

(a) Method 1:

Number of students who wear spectacles

$= 25\%$ of 40

$= \frac{25}{100} \times 40$ express 25% as a fraction

$= 10$

Number of students who do not wear spectacles

$= 40 - 10$

$= 30$

Method 2:

Percentage of students who do not wear spectacles

$= 100\% - 25\%$

$= 75\%$

Number of students who do not wear spectacles

$= 75\%$ of 40

$= \frac{75}{100} \times 40$ expre

$= 30$

(b) Number of girls in the class

$= 40 - 15$

$= 25$

Number of girls with long hair

$= 60\%$ of 25

$= \frac{60}{100} \times 25$ express 60% as a fraction

$= 15$

Proportionality is the relation between 2 quantities that allow a pair of quantities to be obtained from another pair by multiplying with the same factor. So, in Worked Example 3, we can reason out using proportionality:
If 100% relates to 40, then 25% ($100\% \div 4$) relates to $40 \div 4$, i.e. 10. We have, in this case, multiplied the pair of 100% and 40 by $\frac{1}{4}$.

Reflection

Which method do you prefer? Why?

1. There are 50 cards in a set. 18 cards are red while the rest are blue.
   (a) If 30% of the cards have a number printed on them, find the number of cards that do not have a number printed.
   (b) If 75% of the blue cards are laminated, find the number of blue cards that are not laminated.
2. Find the value of each of the following.
   (a) 2.5% of 30 cm (b) $15\frac{3}{4}\%$ of 640 kg (c) 2500% of $4.60

Worked Example 4

Explaining why percentage of quantity is incorrect

Bernard was asked to find the value of 250% of $100 and his answer was $25.
Without performing any calculations, explain to Bernard why his answer is incorrect.

Solution

Since 100% = 1, then 100% of $100 = $100.

$\therefore$ 250% of $100 must be greater than $100, so $25 is incorrect.

Further Questions
Exercise 8A
Questions 10, 15

Waseem was asked to find the value of 3.6% of 1000 kg and his answer was 3600 kg. Without performing any calculations, explain to Waseem why his answer is incorrect.

D. Expressing one quantity as a percentage of another quantity

Expressing two quantities in equivalent forms

1. There are 15 boys and 25 girls in a class.
   (a) (i) Express the number of boys as a percentage of the number of girls.
   Required percentage = ___ (fraction) × 100% = ___ %
   There are ___ % as many boys as girls in the class.
   The following statements are equivalent to the above statement:
   - The number of boys is ___ % of the number of girls.
   - The number of boys is ___ (fraction) of the number of girls.

   (ii) If you were to express the number of girls as a percentage of the number of boys, guess what this percentage will be.
   (iii) Now, express the number of girls as a percentage of that of boys.
   Required percentage = ___ (fraction) × 100% = ___ %
   There are ___ % as many girls as boys in the class.
   The following statements are equivalent to the above statement:
   - The number of girls is ___ % of the number of boys.
   - The number of girls is ___ (fraction) of the number of boys.

   (iv) Is your answer to part (ii) correct?
   If yes, explain to your classmate how you obtained that answer.
   If no, explain to your classmate what your misconception was.

   (b) Given that A and B represent the number of boys and girls in the class respectively, complete Table 8.3.

In words	A is ___ % of B.	B is ___ % of A.
Percentage	A = ___ % × B	B = ___ % × A
Fraction	A = ___ × B	B = ___ × A
Decimal	A = ___ × B	B = ___ × A
Ratio	$A : B$ = ___ : ___	$B : A$ = ___ : ___

Table 8.3

Equivalence
Equivalence expresses the 'equality' of two mathematical objects (in this case, the relationship between A and B). This can be represented in different forms: in words, percentage, fraction, decimal or ratio.

Table 8.3 shows that the relationship between A and B can be expressed in various equivalent forms.

2. Yao Ming is a retired Chinese professional basketball player who played for the Houston Rockets of the National Basketball Association (NBA). In his final season, he was the tallest active player in the NBA, standing at 2.29 m.

   (a) (i) Express your height as a fraction of Yao Ming's height.

   (ii) Express your height to Yao Ming's height as a ratio.

   (iii) Express your height as a percentage of Yao Ming's height.

   (b) In part (a), your height has been expressed as a proportion of Yao Ming's height in three different but **equivalent** forms – fraction, ratio and percentage.
   Discuss which is the most useful form to compare your height with Yao Ming's.

   (c) Express Yao Ming's height as a percentage of your height.
   Based on this percentage, can you visualise how much taller Yao Ming is compared to you?

From the above Class Discussion, we observe that we can use percentage to gauge how large/small a quantity is with respect to another quantity. For example, if your height is about 70% of Yao Ming's height, it means that you are about $\frac{7}{10}$ as tall as him.

Expressing quantity a as a percentage of b

Convert the fraction $\frac{a}{b}$ into a percentage, i.e. $\frac{a}{b} \times 100\%$.

Attention

Both a and b must be of the *same* unit.

Worked Example 5

Expressing one quantity as a percentage of the whole

There are 90 teachers in a school, of which 40 are male. Calculate the percentage of

(i) male teachers, (ii) female teachers,

in the school.

Solution

(i) Percentage of male teachers in the school $= \frac{40}{90} \times 100\%$

$= 44\frac{4}{9}\%$

(ii) **Method 1**

Percentage of female teachers in the school $= 100\% - 44\frac{4}{9}\%$

$= 55\frac{5}{9}\%$

Method 2

Number of female teachers in the school $= 90 - 40$

$= 50$

Percentage of female teachers in the school $= \frac{50}{90} \times 100\%$

$= 55\frac{5}{9}\%$

Reflection

Which method do you prefer? Why?

In a school, there are 450 boys and 750 girls. Calculate the percentage of
(i) boys, (ii) girls,
in the school.

Worked Example

Expressing one quantity as a percentage of another quantity

Ali is 12 years old and his sister is 20 years old. Express
(i) Ali's age as a percentage of his sister's age,
(ii) Ali's sister's age as a percentage of his age.

Solution

(i) Ali's age as a percentage of his sister's age

$$= \frac{\text{Ali's age}}{\text{sister's age}} \times 100\%$$

$$= \frac{12}{20} \times 100\%$$

$$= 60\%$$

(ii) **Method 1:**

Ali's sister's age as a percentage of his age

$$= \frac{\text{sister's age}}{\text{Ali's age}} \times 100\%$$

$$= \frac{20}{12} \times 100\%$$

$$= 166\frac{2}{3}\%$$

Method 2:

From part (i), Ali's age = 60% × sister's age

$$= \frac{60}{100} \times \text{sister's age}$$

Then sister's age $= \frac{100}{60} \times$ Ali's age

∴ Ali's sister's age as a percentage of his age

$$= \frac{100}{60} \times 100\%$$

$$= 166\frac{2}{3}\%$$

Attention

(i) Since Ali's age as a percentage of his sister's age is 60%, we say that
- Ali's age is 60% of his sister's age, or
- Ali's age is 40% (= 100% – 60%) less than his sister's age

Reflection

Which method do you prefer? Why?

Exercise 8A

1. The number of pages in magazine A is 128 and the number of pages in magazine B is 200.
   (i) Express the number of pages in magazine A as a percentage of the number of pages in magazine B.
   (ii) By how many percent is the number of pages in magazine B more than the number of pages in magazine A?
2. Express 1400 ml as a percentage of 2.1 *l*.

Worked Example 7

[illegible]

Li Ting writes the following statement:

"If A is 80% of B, then B is 20% of A."

Explain if you agree with Li Ting's statement.

Solution

If A is 80% of B, then $A = 80\% \times B$

$$= \frac{4}{5}B.$$

So $B = \frac{5}{4}A$

$$= \left(\frac{5}{4} \times 100\%\right) \times A$$

$$= 125\% \text{ of } A$$

$\therefore$ I do not agree with Li Ting's statement.

Problem-solving Tip

We can use a model to help us visualise the comparison.

A is 80% of B. What percentage of A is B?

Practise Now 7

Li Ting writes another statement:

"If A is 20% more than B, then B is 20% less than A."

Explain if you agree with this statement.

Introductory Problem Revisited

In the [illegible], we see that we cannot simply add the percentages of different groups to get the overall percentage. A percentage is a proportion in relation to a whole or another quantity. Hence, the value that a percentage represents depends on what is the whole in the context.

Can you find the number of students in Sindh and Punjab who are enrolled in high schools such that the total percentage obtained is 9.0%? Discuss with your classmates.

E. Comparing two quantities by percentage

Two schools jointly organised a walkathon to raise money for charity.

Since 600 > 400, does it mean that School A had a higher proportion (or percentage) of students who participated in the walkathon compared to School B?

Worked Example 8 shows how a comparison can be done.

Worked Example 8

Comparing two quantities by percentage

School A had 600 out of 1600 students who participated.

School B had 400 out of 1000 students who participated.

Which school had a higher proportion of students who participated in the walkathon?

Solution

Percentage of School A's students who participated

$= \frac{600}{1600} \times 100\%$

$= 37.5\%$

Percentage of School B's students who participated

$= \frac{400}{1000} \times 100\%$

$= 40\%$

$\therefore$ School B had a higher proportion of students who participated in the walkathon.

Attention

Very often, we cannot simply compare the numerical values of two quantities.
Sometimes we have to compare percentages because the bases are different.
In this case, the bases are each school's number of students, which are different.

Practise Now 8

Further Exercise 8A Questions

There were 30 000 people in Village A and 4000 people attended its New Year party.
There were 25 000 people in Village B and 3500 people attended its New Year party.
Which village had a higher proportion of people who attended its party?
Justify your answer with calculations.

Worked Example 9

Comparing data using percentages

Yasir showed his friend his mathematics test results:

Test	Maximum mark	My mark
1	25	18
2	50	35
3	40	25

His friend commented, "You obtained the best result in Test 2!"

(i) Explain why his friend thought that Yasir obtained the best result in Test 2.

(ii) Do you agree with his friend? Justify your reason with calculations.

Solution

(i) His friend thought that Yasir obtained the best result in Test 2 because Yasir obtained the highest mark for Test 2.

(ii) Percentage of marks obtained in Test 1 $= \frac{18}{25} \times 100\%$

$= 72\%$

Percentage of marks obtained in Test 2 $= \frac{35}{50} \times 100\%$

$= 70\%$

$$\text{Percentage of marks obtained in Test 3} = \frac{25}{40} \times 100\%$$
$$= 62.5\%$$

Since the percentage of marks obtained in Test 1 is the highest, then Yasir obtained the best result in Test 1.

$\therefore$ I do not agree with his friend.

The following table shows the amount of sugar found in each type of berry.

Type of berry	Mass of fruit (g)	Amount of sugar (g)
Blackberry	9	0.439
Blueberry	1.5	0.149
Raspberry	1.9	0.084
Strawberry	7	0.326

Cheryl's doctor advises her to reduce her sugar intake.
Advise Cheryl which of the four berries has the lowest sugar content.

Explain if you will accept Li Ting's proposal.

1. What do I already know about percentage that could guide my learning in this section?
2. What are the ways in which I can compare two quantities?

Exercise 8A

1. Express each of the following percentages as a fraction and as a decimal.

(a) 47% (b) 64%

2. Express each of the following as a percentage.

(a) $\frac{4}{5}$ (b) 0.38

3. Express each of the following percentages as a fraction and as a decimal.

(a) $5\frac{1}{8}\%$ (b) 91.4%

4. Express each of the following as a percentage.

(a) $\frac{2}{3}$ (b) 0.776

5. Express each of the following percentages as a fraction/mixed number and as a decimal.

(a) 159% (b) $813\frac{2}{5}\%$ (c) 0.3% (d) 0.014%

6. Express each of the following numbers as a percentage.

(a) $81\frac{2}{3}$ (b) 9.124 (c) 0.0023 (d) $\frac{11}{4000}$

7. There are 120 cars in a multi-storey car park. Given that 30% of them are blue, find the number of cars which are not blue.

8. A company has 12 000 employees. During a downsizing exercise, 2.5% of them were retrenched, 50.75% of them had a pay cut and the rest were unaffected. Find the number of employees who were unaffected by the downsizing.

9. Find the value of each of the following.

(a) 0.8% of 4.5 m

(b) $111\frac{4}{5}\%$ of 24 kg

(c) 312.5% of $70

10. Yasir was asked to find the value of 7.5% of 10 000 m and his answer was 75 000 m. Without performing any calculations, explain to Yasir why his answer is incorrect.

11. Sucrose is a compound produced naturally in plants, from which table sugar is refined. It has the formula $C_{12}H_{22}O_{11}$, which means it is made up of 12 carbon atoms, 22 hydrogen atoms and 11 oxygen atoms. Find the percentage of hydrogen atoms in the compound.

12. For each of the following, express the first quantity as a percentage of the second quantity.

(a) 45 minutes, 1 hour
(b) 25 seconds, 3.5 minutes
(c) 1 year, 4 months
(d) 15 mm, 1 m
(e) 335 cm, 5 m
(f) 1 kg, 800 g
(g) 60°, 360°
(h) 63 cents, $2.10

13. Waseem earns PKR 16 000 per month while Albert earns PKR 68 000 per month. In 2022, Waseem donated a total of PKR 1200 to charitable organisations while Albert donated a total of PKR 4500 to charitable organisations.
Who donated a higher percentage of his annual income to charitable organisations?

14. A class has 40 students, of which 25% are boys. Given that 70% of the boys passed a mathematics test, calculate the number of boys who passed the test.

15. Is x% of 330 or 330% of x greater in value? Explain your answer.

Exercise 8A

16. Ken's monthly salary is $1850. In a particular month, he spent 20.5% of his salary on room rental, $690 on food and $940 on other expenses. Express the amount that he overspent as a percentage of his monthly salary, giving your answer correct to 2 decimal places.

17. There are 600 pages in a novel. Imran reads 150 pages of the novel on Friday and 40% of the remaining pages on Sunday. Express the number of pages that remains to be read as a percentage of the total number of pages in the novel.

18. The maximum number of marks attainable at a Mathematics competition is 60. Cheryl obtains 40 marks, David obtains 46 marks and Vasi obtains 49 marks. The examination board decides that those who score 80% and above will get a gold award, those who score 70% to less than 80% will get a silver award and those who score 60% to less than 70% will get a bronze award. Determine the type of award each student gets.

19. The height of a father is 50% more than that of his son. By how many percent is the height of the son less than that of his father?

20. Bernard, Li Ting and Shaha received PKR 5000, PKR 6000 and PKR 9000 respectively.
    (i) Find the percentage of the sum of money that Li Ting received.
    (ii) Shaha told Bernard, "My share is 180% of yours."
    Explain if you agree with Shaha's statement.

21. Sara wants to hire a secretary. She gives a typing test to the three applicants who apply for the job. The following table shows the results.

Applicant	Number of words typed in three minutes	Number of typo errors
A	214	17
B	252	20
C	229	18

Which applicant is the best typist? Justify your answer with calculations clearly shown.

22. The area of Towns A and B are 15 km² and 24 km² respectively. The space reserved for greenery in Town A is half that in Town B, and the space reserved for greenery in Town B is 25%. Find the percentage of space in Town A that is reserved for greenery.

23. 99 boys and 1 girl are in a lecture theatre. How many boys must leave the theatre so that the percentage of boys becomes 98%?

24. As part of their project, Nadia and Raju conducted a survey on 10% of the number of students in Class A and on 70% of the number of students in Class B.
    (a) Nadia said that 10% + 70%, i.e. 80%, of the total number of students in the two classes had done the survey. Is she correct? Explain your answer.
    (b) Raju, however, said that $\frac{10\% + 70\%}{2}$, i.e. 40%, of the total number of students in the two classes had done the survey. Is he correct? Explain your answer.

Exercise 8A

In 2018, Joyce donated 10% of her annual income to charity. In 2019, she donated 12% of her annual income to charity. When Joyce was commended for being "more charitable in 2019", she said, "I am not more charitable because I donated the same amount in 2018 and in 2019."

(i) Explain how Joyce's statement could be true.

(ii) Given that Joyce's annual income in both years was each between \$30 000 and \$40 000, suggest a possible amount for her annual income for each year.

8.2 Percentage change, percentage point and reverse percentage

A. Percentage change

The change in the value of an item can be expressed as a percentage increase or decrease in the original value.

An **increase** of 5% in the salary of a man who earns \$1600 per month means that for every \$100 of the *original* salary, there is an increase of \$5, i.e. each \$100 in the original salary becomes \$105 in the *new* salary.

$$\therefore \frac{\text{new salary}}{\text{original salary}} = \frac{105}{100}$$

$$\text{New salary} = \frac{105}{100} \times \text{original salary}$$

$$= \frac{105}{100} \times \$1600$$

$$= \$1680$$

Attention
We can also say that the new salary is 105% of the original salary.

$$\text{Alternatively, increase in salary} = \frac{5}{100} \times \$1600$$

$$= \$80$$

$$\therefore \text{new salary} = \$1600 + \$80$$

$$= \$1680$$

On the other hand, a **decrease** of 5% in his salary means that for every \$100 of the *original* salary, there is a decrease of \$5,
i.e. each \$100 in the original salary becomes \$95 in the *new* salary.

$$\therefore \frac{\text{new salary}}{\text{original salary}} = \frac{95}{100}$$

$$\text{New salary} = \frac{95}{100} \times \text{original salary}$$

$$= \frac{95}{100} \times \$1600$$

$$= \$1520$$

Attention
We can also say that the new salary is 95% of the original salary.

Alternatively, decrease in salary $= \frac{5}{100} \times \$1600$

$= \$80$

$\therefore$ new salary $= \$1600 - \$80$

$= \$1520$

Calculating percentage change

(1) Percentage increase/decrease $= \frac{\text{increase / decrease}}{\text{original value}} \times 100\%$

(2) Increase/Decrease = percentage increase/decrease × original value

(3) New value = final percentage × original value

Worked Example 10

Problem involving simple percentage change

The resident population in Country S is made up of citizens and permanent residents. The citizen population was 3.41 million in 2020 and it increased by 0.9% in 2021. Together with the permanent resident population, the resident population increased from 3.93 million in 2020 to 3.97 million in 2021.

(i) Calculate the citizen population in Country S in 2021.

(ii) Calculate the percentage increase in the resident population in Country S from 2020 to 2021.

Solution

(i) Since the citizen population in Country S increased by 0.9% in 2021, its citizen population in 2021 was 100.9% of that in 2020.

$\therefore$ citizen population in Country S in 2021

$=$ final percentage $\times$ original value

$= 100.9\% \times 3.41$ million

$= \frac{100.9}{100} \times 3.41$ million

$= 3.44$ million (to 3 s.f.)

Problem-solving Tip

The original value before any change is always taken as 100%.

(ii) Percentage increase in resident population in Country S from 2020 to 2021

$= \frac{\text{increase}}{\text{original value}} \times 100\%$

$= \frac{3.97 - 3.93}{3.93} \times 100\%$

$= 1.02\%$ (to 3 s.f.)

Practise Now 10

Similar and Further Questions
Exercise 8B
Questions

To discourage water wastage, the cost per CuM of water consumption in Country A is increased based on the volume consumed.

- For households using 0 to 40 CuM of water per month, the cost is increased from \$2.15 per CuM to \$2.44 per CuM.
- For households using more than 40 CuM of water per month, the cost is increased by 22.9% from \$2.61 per CuM.

(i) Calculate the new cost per CuM for households using more than 40 CuM of water per month. Leave your answer to the nearest cent.

(ii) Calculate the percentage increase in the cost per CuM for households using 0 to 40 CuM of water.

Attention

CuM stands for cubic metre. Although we use m^3 for cubic metre in mathematics, CuM is used in the real world in some cases, like utility bills.

Why is it possible to have a 110% increase in the cost of an item but not a 110% decrease in its cost?

Worked Example 11

More complicated problem involving percentage change

The total cost of making a piece of furniture consists of the cost of wood at \$300, the cost of paint at \$200 and wages at \$200. If the costs of wood and paint are increased by 12% and 7% respectively, while wages are decreased by 10%, calculate the percentage increase or decrease in the cost of the furniture.

*Solution

	Original cost	Percentage change	New cost
Wood	\$300	+12%	$\frac{112}{100} \times \$300 = \$336$
Paint	\$200	+7%	$\frac{107}{100} \times \$200 = \$214$
Wages	\$200	–10%	$\frac{90}{100} \times \$200 = \$180$
Furniture	\$700		\$730

$$\text{Percentage increase in the cost of the furniture} = \frac{\text{increase}}{\text{original value}} \times 100\%$$

$$= \frac{\$730 - \$700}{\$700} \times 100\%$$

$$= \frac{\$30}{\$700} \times 100\%$$

$$= 4\frac{2}{7}\%$$

Practise Now 11

Similar and Further Questions
Exercise 8B

The monthly cost of running a small business consists of retail space rental at PKR 700 000, wages at PKR 525 000 and utilities at PKR 140 000. If the retail space rental and wages are decreased by 5% and 6% respectively, while utilities are increased by 7%, find the percentage increase or decrease in the monthly cost of running the business.

Can we add percentages or take average of percentages?

1. In Worked Example 11, explain why we cannot find the change in the cost of the furniture using: $12\% + 7\% + (-10\%) = 9\%$.
2. Yasir was earning a monthly salary of $\$x$ in 2020. In 2021, his salary was increased by 10%. However, in 2022, due to the financial situation of his company, his salary was decreased by 10%. Is it correct to say that his monthly salary in 2022 was $\$x$? Explain your answer.
3. The percentage passes in a Mathematics examination for two Secondary One classes are 80% and 60%. The number of students in each class is 35 and 40 respectively.
   (i) Find the average of the percentage pass by dividing the sum of the percentage passes of the two classes by 2.
   (ii) Find the overall percentage pass by calculating the number of students who passed in each class and expressing this as a percentage of the total number of students.
   (iii) Compare the answers to parts **(i)** and **(ii)**. What do you realise?
   (iv) What lesson can you learn from this question? Is there an exception?

From the above Investigation, we learn that:

In general, we should *not* add percentages or take the average of percentages because the *bases* may be different.

For example, in Question 1 in the Investigation, the bases refer to the original cost of the wood, paint and wages in Worked Example 11, which are different.

In Question 2 in the Investigation, the bases refer to the monthly salaries for 2020 and for 2021, which are different. As a result, the 10% increase in 2021 on the monthly salary for 2020 is not equal to the 10% decrease in 2022 on the monthly salary for 2021.

What do the bases in Question 3 in the Investigation refer to?

Worked Example 12

Percentage change problem involving algebra

The length of a rectangle is twice that of its breadth. If the length of the rectangle is increased by 10% while its breadth is decreased by 10%, determine, if any, the percentage change in its perimeter.

Solution

Let the breadth of the rectangle be x units.

Then the length of the rectangle is $2x$ units.

New length of rectangle $= \frac{110}{100} \times 2x = 2.2x$ units

New breadth of rectangle $= \frac{90}{100} \times x = 0.9x$ units

Original perimeter of rectangle $= 2(2x + x) = 6x$ units

New perimeter of rectangle $= 2(2.2x + 0.9x) = 6.2x$ units

$$\begin{aligned}\text{Percentage increase in perimeter of rectangle} &= \frac{\text{increase}}{\text{original value}} \times 100\% \\ &= \frac{6.2x - 6x}{6x} \times 100\% \\ &= \frac{0.2x}{6x} \times 100\% \\ &= \frac{0.2}{6} \times 100\% \\ &= 3\frac{1}{3}\%\end{aligned}$$

Further
Exercise 8B
Questions 16

1. The length of a rectangle is twice that of its breadth. If the length of the rectangle is increased by 20% while its breadth is decreased by 20%, determine, if any, the percentage change in its perimeter.
2. The length of a rectangle is twice that of its breadth. If the length of the rectangle is increased by 10% while its breadth is decreased by 10%, determine, if any, the percentage change in its area.

B. Percentage point

Percentage point

In this Investigation, we shall explore the difference between 'percentage point' and 'percent'. Read the article in Fig. 8.3 and answer the questions below.

NEWS

Finance Bill: GST to rise from 17% to 18%

The Pakistani government announced measures to raise PKR 170 billion in taxes by June 2023. The measures include increasing the general sales tax (GST) by one percentage point from 17% to 18%.

Fig. 8.3

1. Bernard said that the increase from 17% to 18% change is an increase of 6% and not 1%. How did he arrive at 6%?
2. Do you think Bernard is right in saying that the increase is 6%? Explain your answer.
3. It is mentioned in the passage that the increase from 17% to 18% is one percentage point. What does 'one percentage point' mean and how is it different from one percent (i.e. 1%)? Is the distinction between 'one percentage point' and 'one percent' important? Explain your answer.

We can refer to an increase in other quantities, such as money e.g. \$17 to \$18, as an increase of \$1. This corresponds to a percentage increase of 6%.

However, in the case of percentages, stating an increase from 17% to 18% as a 1% increase may be misintepreted as a percentage increase of 17% by 1%. Thus, in the above Investigation, the increase is described as an increase in **percentage points**.

C. Reverse percentage

In problems involving percentages, we are usually given the whole or what the 100% represents.
In problems involving reverse percentages, we are supposed to find the whole or what the 100% represents

Worked Example 13

Simple reverse percentage problem

In a student council, 30% of the students walk to school. If 48 students walk to school, calculate the number of students in the student council.

Method 1:

$$\begin{aligned}
\div 30 \quad & 30\% \text{ of the students} = 48 & \div 30\\
\times 100 \quad & 1\% \text{ of the students} = \frac{48}{30} & \times 100\\
& 100\% \text{ of the students} = \frac{48}{30}\times 100 \quad \text{------- } (*)\\
& = 160
\end{aligned}$$

$\therefore$ number of students in student council = 160

Method 2:

Let the number of students in the student council be x.

Then $30\% \times x = 48$

$$\begin{aligned}
\frac{30}{100}x &= 48\\
x &= 48 \times \frac{100}{30}\\
&= 160
\end{aligned}$$

$\therefore$ number of students in student council = 160

Big Idea

Proportionality

In [illegible] 1, we observe that when one quantity is divided by 30, the other quantity is also divided by 30; and when one quantity is multiplied by 100, the other quantity is also multiplied by 100. We say that the two quantities vary with each other proportionally.

Reflection

Compare step * in Method 1 and step ** in Method 2.
Do you notice that the two steps are actually the same? In other words, the two methods are related.
Which method do you prefer? Why?

Practise Now 13

Similar and Further Questions

Exercise 8B

70% of the books on a bookshelf are English books. If there are 35 English books on the bookshelf, find the number of books on the bookshelf.

Worked Example 14

Reverse percentage problem involving percentage increase

After an increase of 5%, Joyce's monthly salary becomes \$2205.
Find her original monthly salary.

Solution

Method 1:

After an increase of 5%, Joyce's monthly salary becomes 105% of her original monthly salary.

$$\begin{aligned}105\% \text{ of original monthly salary} &= \$2205\\ 1\% \text{ of original monthly salary} &= \frac{\$2205}{105}\\ 100\% \text{ of original monthly salary} &= \frac{\$2205}{105}\times 100\\ &= \$2100\end{aligned}$$

$\therefore$ Joyce's original monthly salary is \$2100.

Method 2:

Let Joyce's original monthly salary be $\$x$.

$$\begin{aligned}\text{Then } 105\% \times x &= 2205\\ 1.05x &= 2205\\ x &= \frac{2205}{1.05} \quad \text{------} (***)\\ x &= 2100\end{aligned}$$

$\therefore$ Joyce's original monthly salary = \$2100

Reflection

How is step (***) similar to or different from step (**) in Worked Example 13? Which method do you prefer? Why?

Practise Now 14

Similar and Further Questions

Exercise 8B

Questions

1. If the cost of an article is raised by 9% to PKR 141 700, what is the original cost of the article?
2. Every year, the value of an antique vase appreciates by 20% of its value in the previous year. If the value of the vase was \$180 000 in 2020, find its value in 2018.

Worked Example 15

If 6% is deducted from a bill, $282 remains to be paid. How much is the original bill?

Solution

Method 1:

After 6% is deducted from the bill, 94% of the bill remains to be paid.

$$\begin{aligned} 94\% \text{ of the original bill} &= \$282 \\ 1\% \text{ of the original bill} &= \frac{\$282}{94} \\ 100\% \text{ of the original bill} &= \frac{\$282}{94} \times 100 \\ &= \$300 \end{aligned}$$

$\therefore$ original bill = $300

Method 2:

Let the original bill be $\$x$.

$$\begin{aligned} \text{Then } 94\% \times x &= 282 \\ 0.94x &= 282 \\ x &= \frac{282}{0.94} \\ x &= 300 \end{aligned}$$

$\therefore$ the original bill is $300.

Reflection

Which method do you prefer? Why?

1. After a pay cut of 3%, Nadia's monthly salary becomes PKR 722 681. Find her original monthly salary.
2. Every year, the value of a car depreciates by 15% of its value in the previous year. If the value of the car was $86 700 in 2020, find its value in 2018.

1. What new knowledge of percentage have I gained in this section?
2. What is the most confusing part of the concepts taught in this section? How can I overcome it?

Exercise 8B

1. Find the value of each of the following.
   (a) Increase 60 by 35%
   (b) Increase 28 by 125.7%
   (c) Decrease 120 by 45%
   (d) Decrease 216 by $37\frac{1}{2}$%

2. An elastic band which is 72 cm long, is stretched to 90 cm. Find the percentage increase in its length.

3. The price of a desktop computer decreases from PKR 304 300 to PKR 228 225. Find the percentage decrease in its price.

4. A house costs 36% more today than when it was built. If the cost of the house when it was built was $333 000, find its cost today.

5. A car was bought in 2020 for $120 000. In 2021, its value decreased by 20%. In 2022, its value decreased by 10% of its value in 2021. Find the value of the car at the end of 2022.

6. 45% of the students who took part in a creative writing competition were boys. If 135 boys took part in the competition, find the total number of students who took part in the competition.

7. (a) 20% of a number is 17. Find the number.
   (b) 175% of a number is 49. Find the number.
   (c) The result of a number, when increased by 15%, is 161. Find the number.
   (d) The result of a number, when decreased by 20%, is 192. Find the number.

8. In 2018, the Country P stated that the overall number of visitor arrivals increased year on year by 6.2 percent to 17.4 million. Find the number of visitor arrivals in 2017, leaving your answer in millions correct to one decimal place.

9. If 10% is deducted from a bill, PKR 13 050 remains to be paid. How much is the original bill?

10. The number 2400 is first increased by 30%. The value obtained is next decreased by 20%. Find the final number.

11. In 2021, a train carried 8% more passengers than in 2020. In 2022, it carried 8% more passengers than in 2021. Find the percentage increase in the number of train passengers from 2020 to 2022.

12. The production cost of a printer consists of the cost of raw materials at $100, the cost of overheads at $80 and wages at $120. If the costs of raw materials and overheads are increased by 11% and 20% respectively, while wages are decreased by 15%, find the percentage increase or decrease in the production cost of the printer.

13. When the cost of fuel rose by 10%, Albert decreased his fuel consumption by 10%.
    Albert claimed that there was no change in his expenditure on fuel consumption.
    Explain if Albert is right or wrong.

14. The length of a rectangle is twice that of its breadth. If the length of the rectangle is decreased by 10% while its breadth is increased by 10%, determine, if any, the percentage change in its perimeter.

15. The length of a rectangle is increased by 10% while its breadth is decreased by 10%. Determine, if any, the percentage change in its area.

16. Every year, the value of an apartment appreciates by 15% of its value in the previous year.
    If the value of the apartment was $899 300 in 2022, find its value in 2020.

Exercise 3B

17. Every year, the value of a surveying machine depreciates by 25% of its value in the previous year. If the value of the machine was PKR 2 520 000 in 2022, find its value in 2020.

18. The value of an investment portfolio decreased by 8% in 2021. In 2022, its value increased by 5% of its value in 2021. Given that the value of the portfolio at the end of 2022 was \$61 824, find its original value.

19. Ali is 8% taller than Yasir and Vasi is 10% shorter than Yasir. Express Ali's height as a percentage of Vasi's height.

20. The number of girls in a choir is twice that of boys. The music director wants to have the same number of girls and boys in the choir. Find the required percentage increase or decrease in the number of girls if the music director increases the number of boys by

(a) 30%, **(b)** 130%.

21. Shaha is given a choice of two new salary schemes after a promotion.

Scheme A: 12% increase of her current monthly salary

Scheme B: 8% increase of her current monthly salary + PKR 223 800 annual bonus

(a) If Shaha's current monthly salary is PKR 537 000, which of the two schemes is better?

OPEN **(b)** Suggest a possible amount for Shaha's current monthly salary if he were to accept Scheme B because it is a better offer. Support your answer with calculations.

We frequently encounter percentages in supermarkets, shopping malls, schools, newspapers, reports, taxes and many other real-world contexts. Percentage is a powerful way to express the relationship between two quantities. Often, it is used to express the idea of proportionality between two quantities, whereby a pair of quantities can be obtained from another pair by multiplying with the same factor. We also see the importance of understanding the context, and being clear about the quantity that we are referring to when we communicate mathematical statements using percentages.

While percentages are useful in modelling the relationships between quantities in the real world, they can be misinterpreted and misused, especially in fake news. Often, fake news producers like to use sensational headlines such as:

NEWS

More than 60% of the people are unhappy with …

This may give an impression that many people are unhappy when in fact, only 3 people were surveyed, and out of these, 2 were unhappy. This example highlights the importance of taking note of the quantities we are referring to when we interpret percentage statements. In this data-driven world, it is vital that we develop a keen sense for numbers and a strong understanding of percentage to make sense of the huge amount of information around us.

1. x percent (i.e. x%) is defined as x parts per hundred: $x\% = \frac{x}{100}$.
   - Give two examples of how percentage is used in a real-world context.

2. To express one quantity a as a percentage of another quantity b, we convert the fraction $\frac{a}{b}$ into a percentage, i.e. $\frac{a}{b} \times 100\%$.

   Both a and b must be of the *same* unit.
   - Give an example of how a quantity is expressed as a percentage of another.

3. **Percentage change**

   Percentage increase/decrease $= \frac{\text{increase / decrease}}{\text{original value}} \times 100\%$

   Increase/Decrease = percentage increase/decrease × original value

   New value = final percentage × original value
   - Think of one real-life problem involving percentage change, and solve it.

CHAPTER 9

Ratio and Rate

The Vitruvian Man is a famous drawing created by the world-renowned artist Leonardo da Vinci in 1487. The drawing is sometimes called Proportions of Man because it shows the ideal human proportions, which follow the Golden Ratio of $\frac{1+\sqrt{5}}{2}$.

Any object with a geometric proportion that reflects the Golden Ratio, such as a succulent plant, is pleasing to the eye. This ratio also appears in a sea creature called the nautilus. The use of the Golden Ratio can also be seen in many man-made structures, such as the Parthenon and the Great Pyramid.

Learning Outcomes

What will we learn in this chapter?

- What ratio and rate (including speed) are
- How to find ratios involving rational numbers, up to three quantities
- How to distinguish between constant and average rates
- Why ratios and rates have useful applications in real life

Introductory Problem

Table 9.1 shows a utility bill.

Breakdown of Current Charges	Usage	Rate ($)	Amount ($)	Total ($)
Electricity Services				
Reading taken on 02 June 2023	315 kWh	0.2215	69.77	69.77
Gas Services				
Reading taken on 02 June 2023	30 kWh	0.1853	5.56	5.56
Water Services				
Reading taken on 02 June 2023	15.0 CuM	1.19	17.85	
Water Conservation Tax	$17.85	35%	6.25	24.10
Subtotal			99.43	99.43
GST	99.43	18%	17.90	17.90
Current Charges: (inclusive of GST)				**$117.33**

Table 9.1

The 'Breakdown of Current Charges' shows the number of units of water, electricity and gas a household has used in the billing period.

Can you tell how the charges have been calculated?

In this chapter, we will learn more about ratios and rates so that we can figure out how to interpret ratios and rates used in our daily lives, such as those in the above utility bill.

9.1 Ratio

A. Concept of ratio

Ratios can be used to compare two or more numbers.

For example, if there are 30 boys and 10 girls in a class, we say that the ratio of the number of boys to the number of girls is 30 : 10, which we can simplify to 3 : 1.

Attention

The symbol : is called a colon

But in real life, ratios are often used to compare two or more quantities that have units. For example, in a map, 1 cm represents 1 km.

In this case, the two quantities have different units but they are of the *same kind*: length or distance.

A length on a map cannot possibly be equal to the actual distance on the ground, so the ratio cannot be 1
Therefore, we have to convert 1 km to 100 000 cm so that both quantities have the *same unit*.
Then the map scale as a ratio is 1 : 100 000.

- A ratio is a way of comparing two or more quantities of the same kind that either have no units (i.e. just numbers) or are measured in the same unit.
- The ratio $a : b$, where a and b are positive numbers, has no units.

Worked Example 1

Finding ratios

There are 17 boys and 19 girls in a class. Find the ratio of
(i) the number of boys to the number of girls,
(ii) the number of girls to the number of boys.

Solution

(i) Ratio of the number of boys to the number of girls = 17 : 19
(ii) Ratio of the number of girls to the number of boys = 19 : 17

Attention
The order in which a ratio is expressed is important.

Practise Now

There are 33 lemons and 20 pears in a basket. Find the ratio of
(i) the number of lemons to the number of pears,
(ii) the number of pears to the total number of fruits in the basket.

B. Ratio and fraction

Relationship between ratios and fractions

There are 5 green balls and 7 red balls in a bag.
Let A and B represent the number of green balls and red balls respectively.

1. If T is the total number of balls in the bag,

(a) find the ratio of A to T,

$A : T$
$= \quad : \quad$

The ratio of A to T is $\quad : \quad$.

(b) what fraction of T is A, i.e. what is $\frac{A}{T}$?

$\frac{A}{T} = \quad$.

2. **(a)** Find the ratio of A to B.

$A : B$
$= \quad : \quad$

The ratio of A to B is ____ : ____ .
The following statement is equivalent to the above statement.
A is ____ (fraction) of B, i.e. $\frac{A}{B}$ = ____ (fraction).

(b) Find the ratio of B to A.
$B : A$
= ____ : ____

The ratio of B to A is ____ : ____ .
The following statement is equivalent to the above statement.
B is ____ (fraction) of A, i.e. $\frac{B}{A}$ = ____ (fraction).

3. Draw a model to illustrate the relationship between A and B.

4. Work in pairs.
Come up with other scenarios involving two quantities similar to that given above.
Challenge your classmate to write equivalent statements involving ratios and fractions to compare the two quantities. Draw models to illustrate the relationships.

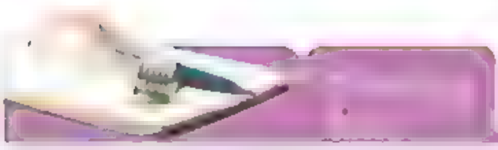
From the above Class Discussion, we learn that:

- The ratio notation $a : b$, where a and b are positive numbers, can be expressed in an equivalent form as a fraction $\frac{a}{b}$.
- Using ratios to compare two quantities of the same unit is equivalent to using fractions to compare the two quantities, e.g. $a : b = 5 : 7$ is equivalent to $\frac{a}{b} = \frac{5}{7}$.

Equivalence

Although $a : b$ is equivalent to $\frac{a}{b}$, $a : b$ is not equal to $\frac{a}{b}$. Therefore, we do *not* write

a b $\frac{a}{b}$



From the above Class Discussion, we have learnt that the ratio notation $a : b$ is equivalent to the fraction $\frac{a}{b}$.

1. Can we have a ratio such as 5 : 0 or 0 : 5? Why or why not?
Hint Think in terms of fractions.

2. The ratio of three numbers or quantities can be represented as $a : b : c$. Can this ratio be expressed as a fraction? Why or why not?

C. Equivalent ratios

We have learnt in Chapter 2 that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{6}{12}$ are **equivalent fractions**.

$$\frac{6}{12} = \frac{2}{4} = \frac{1}{2}$$

Since ratios of two positive numbers be expressed in an **equivalent** form as a fraction, we have:

$$6 : 12 = 2 : 4 = 1 : 2$$

1 : 2, 2 : 4 and 6 : 12 are known as **equivalent ratios**. The *simplest form* is 1 : 2.

Are 23 : 46 and $\frac{1}{3} : \frac{1}{6}$ equivalent to 1 : 2?

$$\begin{aligned} 23 : 46 &= \frac{23}{23} : \frac{46}{23} && \text{divide both parts by 23} \\ &= 1 : 2 \end{aligned}$$

$\therefore$ 23 : 46 is equivalent to 1 : 2.

$$\begin{aligned} \frac{1}{3} : \frac{1}{6} &= \frac{1}{3}\times 6 : \frac{1}{6}\times 6 && \text{multiply both parts by LCM of 3 and 6, i.e. by 6} \\ &= 2 : 1 \end{aligned}$$

$\therefore$ $\frac{1}{3} : \frac{1}{6}$ is not equivalent to 1 : 2.

Information

You can use the key on your calculator to help you express a fraction in its lowest term,

e.g. to express $\frac{6}{12}$ in its lowest term, press 6 1 2 to obtain $\frac{1}{2}$.

Similarly, to simplify a ratio, you can also use the key, e.g. to simplify 6 : 12, press the same sequence of calculator keys and write the answer as 1 : 2.

Equivalent ratios have the same value, e.g. 1 : 2 and 2 : 4. Expressing ratios in the simplest form allows us to check if they are equivalent. Are 23 : 46 and $\frac{1}{3} : \frac{1}{6}$ equivalent?

Example 2

Simplifying ratios

Simplify each of the following.

(a) $1\frac{2}{3} : \frac{5}{6}$ (b) 0.12 : 0.56

Solution

(a)
$$\begin{aligned} 1\frac{2}{3} : \frac{5}{6} &= \frac{5}{3}\times 6 : \frac{5}{6}\times 6 && \text{multiply both parts by 6} \\ &= 10 : 5 \\ &= 2 : 1 && \text{divide both parts by 5 to reduce to simplest form} \end{aligned}$$

(b)
$$\begin{aligned} 0.12 : 0.56 &= 0.12 \times 100 : 0.56 \times 100 && \text{multiply both parts by 100} \\ &= 12 : 56 \\ &= 3 : 14 && \text{divide both parts by 4 to reduce to simplest form} \end{aligned}$$

Exercise 9A

Simplify each of the following.

(a) $2\frac{3}{5} : 1\frac{4}{9}$ (b) 0.36 : 1.2

Worked Example 3

Finding and simplifying ratios of quantities with units

Find the ratio of 600 cm to 1.6 m.

Solution

1.6 m $= 160$ cm   convert to the same unit

$\therefore$ ratio $= 600 : 160$

$= 15 : 4$ divide both parts by 24

Attention

We should *not* write 600 cm : 1.6 m because the colon (i.e. the symbol :) is used to denote a ratio, and a ratio has no units.

So 600 cm : 1.6 m is *not* a ratio.

Exercise 9A

Find the ratio of $2\frac{1}{3}$ kg to 700 g.

Worked Example 4

Finding unknown in ratio

(a) Given that $4x : 9 = 7 : 3$, find the value of x.

(b) If $3 : y = 3^2 : 2^2$, calculate the value of y.

Solution

(a) $4x : 9 = 7 : 3$

$\frac{4x}{9} = \frac{7}{3}$ express ratios as fractions

$4x = 21$

$x = 5\frac{1}{4}$

(b) $3 : y = 3^2 : 2^2$

$y : 3 = 2^2 : 3^2$ change order of parts in each ratio for easier manipulation

$\frac{y}{3} = \frac{4}{9}$ express ratios as fractions

$y = \frac{4}{9} \times 3$

$= 1\frac{1}{3}$

As it is harder to manipulate equality of two ratios, we convert them to the equality of two fractions since the two forms are equivalent.
Thus we see how the conversion from one equivalent form to another can help us to solve problems.

Exercise 9A

1. **(a)** Given that $3a : 7 = 8 : 5$, find the value of a.

   (b) If $5 : b = 5^2 : 4^2$, calculate the value of b.

2. Given that $\frac{3x}{8} = \frac{9y}{4}$, find the ratio of $x : y$.

Worked Example 5

Problem involving ratios of two quantities

The ratio of the number of female participants to the number of male participants at a party is $4 : 9$.

If there are 30 more male participants than female participants, calculate the total number of people who attended the party.

Method 1:

Let the number of female participants = $4x$.

Then the number of male participants = $9x$.

Problem-solving Tip

Draw a model to help you make sense of the problem.

From the model, we form the equation: $9x - 4x = 30$

$$5x = 30$$

$$x = 6$$

Total number of people who attended the party $= (4 + 9) \times 6$

$$= 13 \times 6$$

$$= 78$$

Method 2:

No. of male participants – no. of female participants = 9 parts – 4 parts

= 5 parts

$\therefore$ 5 parts = 30

1 part = 6

Total number of people who attended the party = 13 parts

$= 13 \times 6$

$= 78$

Reflection

How are the two methods similar? Which method do you prefer? Why?

1. The ratio of the number of fiction books to the number of non-fiction books in a library is $5 : 2$. If there are 1421 fiction and non-fiction books altogether, how many more fiction books than non-fiction books are there in the library?
2. Cheryl and Shaha each have a sum of money. The ratio of the amount of money Cheryl has to that of Shaha is $3 : 5$.
   After Shaha gives \$150 to Cheryl, the ratio of the amount of money Cheryl has to that of Shaha becomes $7 : 9$.
   Find the sum of money Cheryl had initially.

D. Ratios involving three quantities

Ratios can also be used to make comparisons among three or more quantities.

For example, if $x = 18$, $y = 27$ and $z = 54$, then $x : y : z = 18 : 27 : 54$

$$= 2 : 3 : 6.$$

From the above, we can deduce that $x : y = 2 : 3$,

$y : z = 3 : 6 = 1 : 2$,

$x : z = 2 : 6 = 1 : 3$.

Attention

The ratio of three quantities can be simplified by multiplying or dividing each term by the same constant, but this ratio cannot be written as a fraction.

Example 6

Ratio involving three quantities

If $x : y = 11 : 8$ and $y : z = 6 : 7$, calculate

(i) $x : y : z$, (ii) $x : z$.

Solution

(i) $x : y = 11 : 8$ $\downarrow \times 3$ $= 33 : 24$

$y : z = 6 : 7$ $\downarrow \times 4$ $= 24 : 28$

$\therefore x : y : z = 33 : 24 : 28$

(ii) From (i), $x : z = 33 : 28$.

Problem-solving Tip

y is the common part in the ratios $x : y$ and $y : z$. To calculate $x : y : z$, we find the equivalent ratios of $x : y$ and $y : z$ such that y has the same number of units in both ratios.

Since $y = 8$ in $x : y$, and $y = 6$ in $y : z$, we will make $y = 24$ because the LCM of 8 and 6 is 24.

Practice Now 6

Similar and Further Questions

Exercise 9A Questions

1. If $x : y = 5 : 6$ and $y : z = 4 : 9$, find
   (i) $x : y : z$, (ii) $x : z$.
2. Find the ratio of 600 m to 0.9 km to 30 000 cm.

Example 7

Problem involving ratios of three quantities

A sum of money is divided among Bernard, Li Ting and Yasir in the ratio 9 : 8 : 7.
After Bernard gives \$25 each to Li Ting and Yasir, the ratio becomes 16 : 17 : 15.
Calculate the amount of money Bernard had at first.

Solution

Let the amount of money Bernard had at first be \$9x.
Then the amount of money Li Ting and Yasir had at first is \$8x and \$7x respectively.

	Bernard	Li Ting	Yasir
Before	\$9x	\$8x	\$7x
After	\$$(9x - 50)$	\$$(8x + 25)$	\$$(7x + 25)$

$$\therefore \quad \frac{9x-50}{8x+25} = \frac{16}{17}$$
$$17(9x - 50) = 16(8x + 25)$$
$$153x - 850 = 128x + 400$$
$$153x - 128x = 400 + 850$$
$$25x = 1250$$
$$x = 50$$

$\therefore$ amount of money Bernard had at first $= 9 \times \$50$
$= \$450$

Problem-solving Tip

You can also find the value of x by solving $\frac{9x-50}{7x+25} = \frac{16}{15}$ or $\frac{8x+25}{7x+25} = \frac{17}{15}$.

A sum of money is divided among Raju, Albert and Vasi in the ratio 6 : 4 : 5.
After Raju gives \$30 to Albert and \$15 to Vasi, the ratio becomes 7 : 6 : 7.
Find the amount of money Raju had at first.

E. Ratios in real-world contexts

Let us look at some ratios we encounter in the real world.

Making sense of ratios used in real-world contexts

1. The aspect ratio of a television screen refers to the ratio of its width to its height. The standard aspect ratio is 4 : 3 while the widescreen aspect ratio is 16 : 9.
   When an image with a 4 : 3 aspect ratio is displayed on a screen with a 16 : 9 aspect ratio, the image becomes distorted, as shown in Fig. 9.1.

4 : 3 image on a 4 : 3 screen — 4 : 3 image on a 16 : 9 screen

Fig. 9.1

   Explain why this happens.

2. The instruction on the label on a bottle of drink concentrate reads: "Mix 1 part of concentrate to 4 parts of water".
   - **(a)** **(i)** Explain what this instruction means.
     - **(ii)** Why is the term "parts" used instead of specifying an amount, e.g. 50 ml of concentrate to 200 ml of water?
   - **(b)** Write down the following.
     - **(i)** The ratio of concentrate to water.
     - **(ii)** The percentage of concentrate in the drink made.
     - **(iii)** The fraction of water in the drink made.
   - **(c)** If you would like to make a drink that has twice as much concentrate compared to the given ratio of 1 : 4, what would the new ratio be?
   - **(d)** Comparing a drink with a 3 : 7 ratio of concentrate to water to that with a 1 : 2 ratio, which drink would have a stronger flavour? Explain why.

Worked Example 8

Ratio in real-world contexts

Joyce's class sold orange juice at a school carnival. To prepare the orange juice, her classmates mixed 5 litres of orange syrup with water, according to a dilution ratio of 1 part syrup to 3 parts water. The orange juice was sold at $1 for a 200-ml cup.

(i) Calculate the amount they collected if all the orange juice was sold out.

(ii) After the carnival, Joyce commented, "We could have doubled the amount collected during the sale by mixing the orange syrup with water in the ratio 1 : 6."
Is Joyce correct? Explain.

*Solution

(i) Amount of orange juice made for sale $= (1 + 3) \times 5$
$= 20$ litres

Amount of money collected from sale $= \dfrac{20 \times 1000}{200} \times \$1$
$= \$100$

(ii) Suppose Joyce's class made orange juice in the ratio 1 : 6.
Amount of orange juice made for sale $= (1 + 6) \times 5$
$= 35$ litres

Since mixing 5 litres of orange syrup with water in the ratio 1 : 6 does not produce twice the amount of orange juice obtained from mixing 5 litres of orange syrup with water in the ratio 1 : 3, the amount collected would not have doubled.

Practice Now 8

Similar and Further Questions
Exercise 9A
Questions 22(a), (b)

Li Ting wants to buy 500-ml bottles of ultra-concentrated washing detergent that suggest a mixing ratio of 1 part detergent to 4 parts water to make the washing solution.

(i) If Li Ting needs to make 5 litres of washing solution, find the number of bottles of ultra-concentrated washing detergent she has to buy.

(ii) Using the number of bottles bought in part (i), Li Ting decides to make a more diluted washing solution than suggested. If she makes 10 litres of washing solution, what is the mixing ratio she uses instead?

Exercise 9A

1. There are 14 boys and 25 girls in a school badminton team. Find the ratio of
   (i) the number of boys to the number of girls,
   (ii) the number of girls to the total number of players in the team.

2. Simplify each of the following ratios.
   (a) $\frac{3}{8} : \frac{9}{4}$ (b) $1 : \frac{3}{7}$
   (c) $0.45 : 0.85$ (d) $1.6 : 4$

3. Find the ratio of
   (a) 1.5 m to 300 cm, (b) 600 ml to 1.2 *l*,
   (c) 50¢ to $1.25, (d) 2.4 kg to 4000 g.

4. (a) Find the value of a if $a : 400 = 6 : 25$.
   (b) Given that $4^2 : 3^2 = 8 : 3b$, find the value of b.

5. A certain amount of money is shared between Waseem and Ken in the ratio 5 : 9.
   If Waseem gets $44 less than Ken, find the total amount of money that is shared between the two boys.

6. Given that $a : b : c = 75 : 120 : 132$,
   (i) simplify $a : b : c$,
   (ii) find $b : a$,
   (iii) find $b : c$.

7. Simplify each of the following ratios.
   (a) $\frac{2}{3} : \frac{3}{2} : \frac{5}{8}$ (b) $2 : \frac{7}{6} : \frac{7}{9}$
   (c) $0.33 : 0.63 : 1.8$ (d) $1.4 : 7 : 6.3$

8. Find the ratio of
   (a) 580 ml to 1.12 *l* to 104 ml,
   (b) 2.8 kg to 700 g to 1.05 kg,
   (c) 32 m to 2.4 km to 64.8 m,
   (d) $7.60 to 84¢ to $6.

9. Nadia, Yasir and David make a total of 1530 toys in the ratio 12 : 16 : 17. Find
   (i) the number of toys Yasir makes,
   (ii) the amount of money David earns if he is paid $1.65 for each toy.

10. In a particular year, a total of 510 000 patients were admitted to hospitals. There were 12 000 doctors and 38 000 nurses registered in the country, and 800 volunteers who helped out in the hospitals. Find the ratio of
    (i) the number of patients to the number of doctors,
    (ii) the number of doctors to the number of nurses to the number of volunteers.

11. Simplify each of the following ratios.
    (a) $0.75 : 3\frac{5}{16}$ (b) $7\frac{1}{7} : 4.5$
    (c) $24\% : 1\frac{1}{5}$ (d) $0.84 : 0.84\%$

12. Find the ratio of
    (a) 2.475 m to 33 cm, (b) $4\frac{1}{5}$ kg to 61.6 g,
    (c) $3\frac{3}{4}$ *l* to 250 ml, (d) $2.05 to 75¢.

13. (a) Given that $\frac{2x}{5} = \frac{3y}{8}$, find the ratio of $x : y$.
    (b) Given that $\frac{7x}{9} = \frac{14y}{3}$, find the ratio of $x : y$.

14. In a school of 1200 students, the ratio of the number of teachers to students is 1 : 15.
    After some teachers join the school, the ratio of the number of teachers to students becomes 3 : 40. Find
    (i) the initial number of teachers in the school,
    (ii) the number of teachers who join the school.

Exercise 9A

15. If $p : q = \frac{3}{4} : 2$ and $p : r = \frac{1}{3} : \frac{1}{2}$, find

(i) $p : q : r$, (ii) $q : r$.

16. Given that $x : 3 : \frac{9}{2} = \frac{15}{4} : 4\frac{1}{2} : y$, find the value of x and of y.

17. The numbers of roses, sunflowers and tulips a florist has are in the ratio 5 : 6 : 9.
After the florist sells 50 roses, the ratio becomes 3 : 4 : 6.
Find the number of roses left after 50 of them are sold.

18. Vasi makes a fruit drink for a party. He uses lemonade, strawberry syrup and carbonated water in the ratio 3 : 1 : 4. He has enough lemonade and carbonated water to make 6 litres of the fruit drink.

(i) Strawberry syrup is sold in 240-ml bottles. Find the number of bottles of strawberry syrup he has to buy.

(ii) Vasi decides to use up all the strawberry syrup bought as calculated in part (i). Find the additional amounts of lemonade and carbonated water Vasi has to buy in order to maintain the ratio of 3 : 1 : 4.

19. Find the number that must be added to 3 and 8 so that the ratio of the first number to the second number becomes 2 : 3.

20. Given that $x : y = 3 : 4$ and $y : z = 5 : 8$, find the value of $\frac{2y}{3x - y + 2z}$.

21. OPEN Three numbers x, y and z are such that $x : y : z = 5 : 4 : 3$.

(a) Suggest one set of values of x, y and z if $x \neq 5$.

(b) Suggest one set of values of x, y and z if $0 < x < 1$.

22. Bernard and Shaha are competing in an election to be the president of the Student Council.
The ratio of votes Bernard receives to that which Shaha receives from Class A is 1 : 2.
The ratio of votes they receive from Class B is 3 : 2.

(a) Bernard claims that neither he nor Shaha wins the election because the ratio of votes he receives to that Shaha receives is 4 : 4.
Is Bernard correct? Explain

OPEN (b) Shaha claims that she has won the election. Suggest one set of the number of votes each of them received from each class such that Shaha's claim is true.

9.2 Rate

A. Concept of rate

In Section 9.1, we have learnt that a *ratio* is used to compare two or more quantities of the same kind that either have no units (i.e. just numbers) or are measured in the same unit.

For example, the ratio of the number of boys to the number of girls in a particular team is 3 : 1.

We can also say that for every girl in the team, there are 3 boys, or there are 3 boys *per* girl in the team.

Information

The word "*per*" means "for each

If there are now 2 girls in the team, we see that there will be 6 boys

The statement "3 boys per girl" is called a **rate**. Rate tells us how one quantity *changes* with another quantity. In this section, we will learn more about rates, and the relationship between ratio and rate.

Different types of rates

Part 1: Typing speed

An average person types 40 words in one minute.

We can express the typing speed as a rate of 40 words *per* minute.

The typing speed or rate measures how the number of words typed *changes* with the duration.

(a) What two quantities does the typing rate compare?

(b) Are the two quantities of the same kind or of different kinds? How can you tell?

(c) Give another example of a rate that compares two quantities of *different kinds*.

In this case, the typing speed or rate is expressed as the number of units of one quantity **per unit** of another quantity.

Part 2: Speaking rate

Search the Internet for the video "Math Snacks: Bad Date".

During the dinner, the man spoke 175 words but the woman only spoke 25 words.

The woman complained to her friend on the phone, "For every word I spoke, he spoke 7 words."

In this example, the rate is the number of words the man spoke per word that the woman spoke, i.e. 7 words spoken by the man *per* word spoken by the woman.

(d) What two quantities was the woman comparing?

(e) Are the two quantities of the same kind or of different kinds? How can you tell?

(f) Give another example of a rate that compares two quantities of the *same kind*.

Part 3: Relationship between ratio and rate

Although ratio and rate are two different **measures** used to compare quantities, there is an overlap when the comparison is between two quantities of the same kind.

For instance, at the start of Section 9.2, the ratio of the number of boys to the number of girls, 3 : 1, can be converted to a rate of 3 boys per girl in the class.

(g) Find another example of a ratio in Section 9.1 and convert it to a rate.

A rate that compares two quantities of the same kind can also be converted to a ratio.

For example, in **Part 2**, the *rate* is 7 words spoken by the man per word spoken by the woman.

(h) Express the number of words spoken by the man to the number of words spoken by the woman as a *ratio*.

Part 4: Cost per 100 grams

Let us now calculate the cost of coffee powder for the same amount of coffee for each of the two options.

> **Problem-solving Tip**
> To find the cost per 100 g, divide $5.80 by 2 for Option A, and divide $7.45 by 2.5 for Option B. Why?

Option A

200 g Coffee powder
$5.80

Option B

200 g + 50 g [illegible]

(i) Do you prefer to calculate the cost per gram of coffee powder or the cost per 100 g of coffee powder? Why?

Calculating the cost per 100 g of coffee powder will give us values that are easier to manage and compare.

The cost per 100 g of coffee powder is a rate.
In other words, a rate can be expressed as the number of units of one quantity **per n units** of another quantity, where n is a positive integer.

(j) Give another example of a rate that is expressed as the number of units of one quantity per n units of another quantity, where $n \neq 1$ (the two quantities can be of the same kind or of different kinds).

Part 5: Postage rates

Table 9.2 shows the postage rates in a particular country.

> **Attention**
> The second quantity in a rate does not have to be time, e.g. postage rates. Think of other examples where neither quantity is time.

Type of mailer	Weight up to and including	Postage Cost
Letters	100 g	$1.08
Large letters	100 g	$1.85
	250 g	$2.60
	500 g	$3.38
	750 g	$4.20
Small parcels	2 kg	$5.67
Medium parcels	2 kg	$1.08
	10 kg	$8.85
	20 kg	$16.49

Table 9.2

(k) How much do you have to pay if you are posting

- a large letter that weighs 280 g,
- a medium parcel that weighs 10 kg,
- a small parcel that weighs 1.8 kg,
- a letter that weighs 120 g?

The postage rates are not expressed as the cost per gram or per 10 g, but in terms of weight *categories* or brackets. Another example of a rate that uses *brackets* is the income tax, which we will see in Book 2.

From the Class Discussion on pages 217 and 218, we learn that:

- A **rate** is a way of comparing how one quantity [illegible] with another quantity. The two quantities can be of the same kind or of different kinds.
- A rate that compares two quantities of the same kind can be converted to a ratio and vice versa. We use rate when comparing how one quantity changes with another quantity Otherwise, we use ratio.
- A rate can be expressed as the number of units of one quantity [illegible] of another quantity, where n is a positive integer, or it can be expressed in terms of categories or brackets, depending on the context.

Worked Example 9

Rates involving quantities of different kinds

Shop A

6 eggs cost $1.50

Shop B

12 eggs cost $2.40

Shop A sells eggs at $1.50 per half dozen whereas Shop B sells eggs of the same size and quality at $2.40 per dozen. Which shop should we buy the eggs from?

Solution

Method 1

Shop A:

6 eggs cost $1.50

$\div 6$

1 egg costs $\dfrac{\$1.50}{6}$

$= \$0.25$

Shop B:

12 eggs cost $2.40

$\div 12$

1 egg costs $\dfrac{\$2.40}{12}$

$= \$0.20$

$\therefore$ we should buy the eggs from Shop B.

Method 2

Shop A: Price $= \dfrac{\$1.50}{6} = \$0.25$ per egg

Shop B: Price $= \dfrac{\$2.40}{12} = \$0.20$ per egg

$\therefore$ we should buy the eggs from Shop B.

Proportionality

In Worked Example 9, two quantities (the number of eggs and their cost) vary proportionally. When one quantity is divided by 6 (or multiplied by $\frac{1}{6}$), we can also divide the other quantity by 6. We say that the cost of the eggs is directly proportional to the number of eggs. We also note that when two quantities are proportional to each other, the rate of change of one quantity with respect to the other quantity is a constant. So, in Method 2, we can also use this constant rate for comparison.

Reflection

A shorter method to solve this problem is to compare the cost of 6 or 12 eggs.
Will the shorter method always work? Why or why not?

Practice Now 9

Li Ting can type 720 words in 16 minutes, Vasi can type 828 words in 18 minutes and Ali can type 798 words in 19 minutes. Who is the fastest typist?

10

Rates involving quantities of same kind

The unemployment rate measures the number of unemployed people against the number of working people.
It is calculated by dividing the number of unemployed people by the number of working people, and then expressed as a percentage.
In 2022, the unemployment rate in Pakistan was approximately 6.42%.
Explain what an unemployment rate of 6.42% means.

Attention
It is better to express a rate as a percentage when the rate is small.
Writing the rate as 0.0642 is more difficult to read.

Explanation 1:
An unemployment rate of 6.42% means that there are 6.42 unemployed people per 100 working people.

Explanation 2:
An unemployment rate of 6.42% means that there are 642 unemployed people for every 10 000 working people.

Attention
The two explanations are slightly different ways of expressing the same thing. Can you express it in another way?

Exercise 9B
Question

The global literacy rate for all people in 2020 was 86.3%. The literacy rate is calculated by dividing the number of people who are literate by the total population aged 15 years and over, and then expressed as a percentage. Explain what a literacy rate of 86.3% means.

11

Rate of change of *A* with *B* versus rate of change of *B* with *A*

Sara comes across two deals for the body wash she wants to buy. Which option gives the better value for money?

Option A	Option B
950 ml Body wash	950 ml + 250 ml Body wash
$6.95	$7.95

Method 1:
Option A: Cost per 100 ml of body wash

$$= \frac{\$6.95}{9.5}$$
$$= \$0.73 \text{ (to the nearest cent)}$$

Problem-solving Tip
Although rates can be calculated as 'cost per 100 ml' or 'cost per ml', why do we calculate the rates in Method 1 as 'cost per 100 ml' instead of 'cost per ml'?

Option B: Cost per 100 ml of body wash

$$= \frac{\$7.95}{9.5 + 2.5}$$

$= \$0.66$ (to the nearest cent)

Since the cost per 100 ml for Option B is less than that for Option A, Option B gives the better value for money.

Option A: Amount of body wash per dollar $= \frac{950}{6.95}$

$= 137$ ml (to 3 s.f.)

Option B: Amount of body wash per dollar $= \frac{950 + 250}{7.95}$

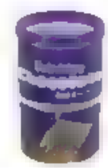
$= 151$ ml (to 3 s.f.)

Since the amount of body wash per dollar for Option B is more than that for Option A, Option B gives the better value for money.

Attention

We can compare two rates by calculating the number of units of quantity A per n units of quantity B, or the number of units of quantity B per n units of quantity A.
But the basis of comparison is different.

A 900-g tin of chocolate drink powder costs PKR 3870 and a 1.45-kg tin of the same brand of chocolate drink powder costs PKR 5800. Which tin gives the better value for money?



900 g for PKR 3870

1.45 kg for PKR 5800

Looking back at Worked Example 11,

(i) can the answer be found using ratios?

(ii) what is the relationship between ratio and rate in this case?

Introductory Problem Revisited

After learning about rates, can you use the concepts learnt to verify the calculations shown in the utility bill in the Introductory Problem? Discuss this with your classmates.

Take a look at your household utility bill and explain to your parents how it is calculated.

B. Average rates and constant rates

Average pulse rate

1. Take your pulse for 1 minute. Record your reading in Table 9.3.
2. Repeat Step 1 twice.

	First reading	Second reading	Third reading
Pulse rate (per minute)			

Table 9.3

3. Are the three pulse rates in Table 9.3 equal? Explain your answer.
   (If all your pulse rates in Table 9.3 happen to be equal, check one of your classmates' readings. Are all his/her pulse rates equal? Why or why not?)
4. Find your **average** pulse rate per minute.
5. The three pulse rates in Table 9.3 are usually not equal because your heart does not beat at a constant rate. Take a look at your first reading. Do you think your heart was beating at a constant rate within that one minute? Explain your answer.
6. Give another real-life example of a rate that is not constant.
7. An example of a **constant rate** is the hourly wage of working in a café. If the hourly wage for working in a café is \$6, we will earn \$12 if we work for 2 hours, \$18 if we work for 3 hours, and so on.
   Give another real-life example of a rate that is constant.

Worked Example 12

Applications of constant and average rates

(a) A shopping centre charges a fixed amount per minute of parking. Raju pays \$6 for parking his car in the shopping centre for 2.5 hours.
Calculate how much he pays to park his car in the shopping centre for $1\frac{3}{4}$ hours.

(b) On another occasion, Raju drives from City P to City Q. His car requires 18 litres of petrol to travel a distance of 243 km.

(i) How far, on average, can his car travel on 44 litres of petrol?

(ii) Given that the cost of petrol is \$1.87 per litre, how much, on average, does he have to pay for the petrol to travel a distance of 675 km?

Reflection

Are the rates in **(a)** and **(b)** constant or average? Explain your answer.

Solution

(a) **Method 1:**

2.5 hr of parking cost \$6

$\div 2.5$: 1 hr of parking costs $\frac{\$6}{2.5}$

$\times 1\frac{3}{4}$: $1\frac{3}{4}$ hr of parking cost $\frac{\$6}{2.5} \times 1\frac{3}{4}$

$= \$4.20$

$\therefore$ amount that Raju will have to pay = \$4.20

Method 2:

Cost of parking per hour $= \frac{\$6}{2.5}$

$= \$2.40$

$\therefore$ amount that Raju will have to pay $= \$2.40 \times 1\frac{3}{4}$

$= \$4.20$

In Worked Example 12, the parking cost is proportionally related to the duration of parking. We observe that when one quantity is divided by 2.5, the other quantity is also divided by 2.5; and when one quantity is multiplied by $1\frac{3}{4}$, the other quantity is also multiplied by $1\frac{3}{4}$. In this proportional relationship, the parking rate is a constant and so we can use it in Method 2 to solve our problem.

(b) **(i)** Average distance travelled on 1 litre of petrol $= \frac{243}{18}$

$= 13.5$ km

Average distance travelled on 44 litres of petrol

$= 13.5 \times 44$

$= 594$ km

(ii) Average amount of petrol required to travel a distance of 675 km $= \frac{675}{13.5}$

$= 50$ litres

Average amount that Raju will have to pay $= 50 \times \$1.87$

$= \$93.50$

Practise Now 12

Exercise 9B

1. **(a)** A bus company charges \$2.70 per kilometre to ferry 36 children for an outing. The accompanying teacher travels for free. What is the cost per child if the distance travelled for the trip is 32.5 km?

(b) A car uses 25 litres of petrol to travel 265 km. Find, on average,

(i) the distance that the car can travel on 58 litres of petrol,

(ii) the amount that the car owner has to pay to travel 1007 km if a litre of petrol costs \$1.95.

2. In a competition, 5 people can eat 20 steamed buns in 3 minutes 20 seconds. Assuming that everyone consumes steamed buns at the same rate and that the rate of consumption remains constant throughout the competition, find the number of steamed buns 10 people can eat in 5 minutes.

Worked Example 13

Currency exchange

(a) The rates of exchange between the Australian dollar (AUD), Korean won (KRW) and Pakistani rupee (PKR) are AUD 1 = PKR 194.3272 and KRW 1000 = PKR 228.0131. Convert

(i) AUD 543, **(ii)** KRW 48 500,

into PKR, giving your answers correct to the nearest PKR.

(b) The rates of exchange between the pound sterling (GBP), Japanese yen (JPY) and Pakistani rupee (PKR) are GBP 1 = PKR 381.2773 and JPY 100 = PKR 207.0427. Convert

(i) PKR 6800 into GBP, **(ii)** PKR 8450 into JPY,

giving your answers correct to the nearest unit of foreign currency.

Information

Currency symbols are used to denote currencies, such as \$ (dollar) and ฿ (baht)

In 1978, three-letter [illegible] were introduced to standardise the different currencies used, e.g. PKR (Pakistani rupee), AUD (Australian dollar) and THB (Thai baht).

Solution

(a) (i) AUD 1 = PKR 194.3272

AUD 543 = PKR 194.3272 × 543

= PKR 105 520 (to the nearest PKR 1)

(ii) KRW 1000 = PKR 228.0131

$$\text{KRW } 1 = \text{PKR } \frac{228.0131}{1000}$$

$$\text{KRW } 48\,500 = \text{PKR } \frac{228.0131}{1000} \times 48\,500$$

= PKR 11 059 (to the nearest KRW 1)

(b) (i) PKR 381.2773 = GBP 1

$$\text{PKR } 1 = \text{GBP } \frac{1}{381.2773}$$

$$\text{PKR } 6800 = \text{GBP } \frac{1}{381.2773} \times 6800$$

= GBP 17.83 (to the nearest GBP 0.01)

(ii) PKR 207.0427 = JPY 100

$$\text{PKR } 1 = \text{JPY } \frac{100}{207.0427}$$

$$\text{PKR } 8450 = \text{JPY } \frac{100}{207.0427} \times 8450$$

= JPY 4081 (to the nearest JPY 1)

Reflection

Did you realise that the exchange rates are all given to more than 3 significant figures? What would happen if the rates were given to only 3 significant figures?

1. **(a)** The rates of exchange between the New Zealand dollar (NZD), Philippines peso (PHP) and Pakistani rupee (PKR) are NZD 1 = PKR 178.7524 and PHP 100 = PKR 535.6015. Convert
   (i) NZD 2360,
   (ii) PHP 25 600,
   into PKR, giving your answer correct to the nearest PKR.

   (b) The rates of exchange between the euro (EUR), Thai baht (THB) and Pakistani rupee (PKR) are EUR 1 = PKR 324.9140 and THB 100 = PKR 768.5302. Convert
   (i) PKR 5690 into EUR, **(ii)** PKR 7460 into THB,
   giving your answers correct to the nearest unit of foreign currency.

2. A family travels from Hong Kong to Singapore for a holiday. They exchange H[illegible]00 to Singapore dollars at an exchange rate of HK\$100 = S\$16.988. They spend a total of S\$3500 in Singapore and convert the remaining Singapore dollars into Hong Kong dollars at the end of the trip at an exchange rate of HK\$100 = S\$16.995.
Find the amount of Hong Kong dollars they receive, giving your answer correct to the nearest dollar.

Intermediate Basic

Exercise

1. (a) A typist types 1800 words in 1 hour. Find the number of words that she can type per minute.
   (b) If \$120.99 is charged for 654 units of electricity used, find the cost of one unit of electricity.
   (c) A man pays a total of \$4800 for flat rental in 3 months. Find his monthly rental rate.
   (d) If the mass of a metal bar which is 3.25 m long is 15 kg, find its mass per metre.

2. Ali folded 15 paper planes in 20 minutes, Nadia folded 18 paper planes in 25 minutes and Kumar folded 16 paper planes in 21 minutes. Who folded the paper planes at the fastest rate?

3. The crime rate measures the number of crimes recorded against the total population in a country. It is calculated by dividing the number of crimes recorded by the total population in the country, and then expressed as a percentage. The crime rate for violent and serious property crimes in a particular country in 2022 was 0.004%. Explain what the rate of 0.004% means.

4. For each ornament that a worker makes, he is paid \$1.15. He makes 4 ornaments every 15 minutes. Find the amount earned by the worker if he works for 3 hours.

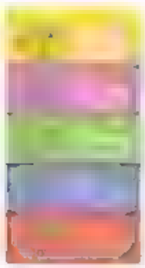
5. A car requires 22 litres of petrol to travel a distance of 259.6 km. Find
   (i) the distance that the car can travel on 63 litres of petrol,
   (ii) the amount that the car owner has to pay to travel a distance of 2013.2 km if a litre of petrol costs \$1.99.

6. 200 g of fertiliser is required for a plot of land that has an area of 8 m^2. Find
   (i) the amount of fertiliser needed for a plot of land that has an area of 14 m^2,
   (ii) the area of land that can be fertilised by 450 g of fertiliser.

7. The rates of exchange between the American dollar (USD), Indonesian rupiah (IDR) and Pakistani rupee (PKR) are USD 1 = PKR 295.1922 and IDR 100 = PKR 1.9201. Convert
   (i) USD 765, (ii) IDR 2 560 000,
   into PKR, giving your answers correct to the nearest PKR.

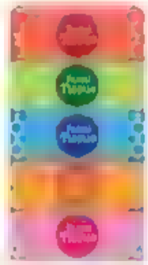
Exercise 9B

8. Sara compares the prices of some facial tissues she intends to buy.

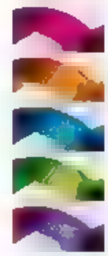


Brand A facial tissue
5 × 200 per pack
Buy 2 and get $1.05 off
$5.75

Brand B facial tissue
5 × 150 per pack
$3.95



Brand C facial tissue
5 × 200 per pack
$4.45



Brand D facial tissue
5 × 200 per pack
Buy 2 and get $1.75 off
$5.75

Help Sara determine which brand is the best buy.

9. A piece of metal is heated to 428 °C before it is left to cool. The temperature of the metal falls at a rate of 23 °C per minute for the first 3 minutes, at a rate of 15 °C per minute for the next 15 minutes and then at a rate of 8 °C per minute until it reaches room temperature of 25 °C. Find

(i) the temperature of the metal after 9 minutes,

(ii) the total time needed for the metal to reach a temperature of 25 °C from 428 °C.

10. A cook uses 15 2-litre bottles of cooking oil in 4 weeks. If he decides to buy 5-litre tins of cooking oil instead, how many tins of cooking oil will he use over a 10-week period if the rate at which he uses it remains unchanged?

11. 224 hours are required to complete a project. 4 men are employed for this project.

(i) The hourly rate of each man is $7.50. Find the total amount to be paid to the men.

(ii) Their normal working hours are from 9 a.m. to 6 p.m., with a one-hour lunch break. Given that their overtime rate is 1.5 times their hourly rate, find the total amount to be paid to the 4 men if the project is to be completed in 4 days.

12. A couple travels from New Zealand to Singapore for a holiday. They exchange NZ$3200 to Singapore dollars at a rate of NZ$100 = S$94.85. They spend a total of S$2560.20 in Singapore and convert the remaining Singapore dollars into New Zealand dollars at the end of the trip at a rate of NZ$100 = S$97.65. Find the amount of New Zealand dollars they receive, giving your answer correct to the nearest cent.

13. Money changers usually display exchange rates in a table form. There are two columns in the table with the headers '*We Sell*' and '*We Buy*'. The column '*We Sell*' refers to the money changer selling 1 unit of foreign currency for the local currency. '*We Buy*' refers to the money changer buying 1 unit of foreign currency from the customer and giving him/her the local currency.
Below are some exchange rates offered by two money changers in Pakistan.

PQ Money Changer

Currency	We Sell	We Buy
United States dollar	308.3304	303.8489
New Zealand dollar	204.4186	197.4825

Exercise 9B

RS Forex

Currency	We Sell	We Buy
United States dollar	308.2204	304.1395
New Zealand dollar	203.6634	198.2387

(i) Vasi is leaving for New Zealand and he wants to exchange some Pakistani rupees for New Zealand dollars. Which money changer would you recommend to him?

(ii) Raju just returned from Boston and he wants to exchange his excess United States dollars for Pakistani rupees. Which money changer would you recommend to him?

11. A watch was priced at GBP 430 on website G, based in Germany and at US$600 on website U, based in the United States. Sara lives in Pakistan and wishes to purchase the watch from either of these websites. She found that the exchange rates are PKR 100 = GBP 0.2650 and PKR 100 = USD 0.3299.

(i) Which website offers a better buy? Explain your answer.

(ii) Both websites charge delivery fees to send the watch to Pakistan. Find the maximum difference in the delivery fees so that the website in part (i) remains a better buy.

9.3 Speed

In this section, we will learn about speed, which is a special type of rate. Let us first discuss how to calculate duration in different units of time.

A. Duration

In primary school, we learnt about various units of measurement for time, such as seconds, minutes and hours. We have also learnt to tell time using a.m. and p.m., in which the day is divided into two cycles of 12 hours each. This is known as the 12-hour clock format.

Another time convention is the 24-hour clock, which counts the day over a continuous period of 24 hours.

Information

The abbreviation 'a.m.' stands for [illegible], which means "before midday" in Latin. 'p.m.' stands for *post meridiem*, translating to "after midday".

In Fig. 9.2(a), some hours of the day in the 12-hour format and the corresponding time in the 24-hour form[illegible] a[illegible] shown. What are the missing times in the figure? Fill in the blanks.

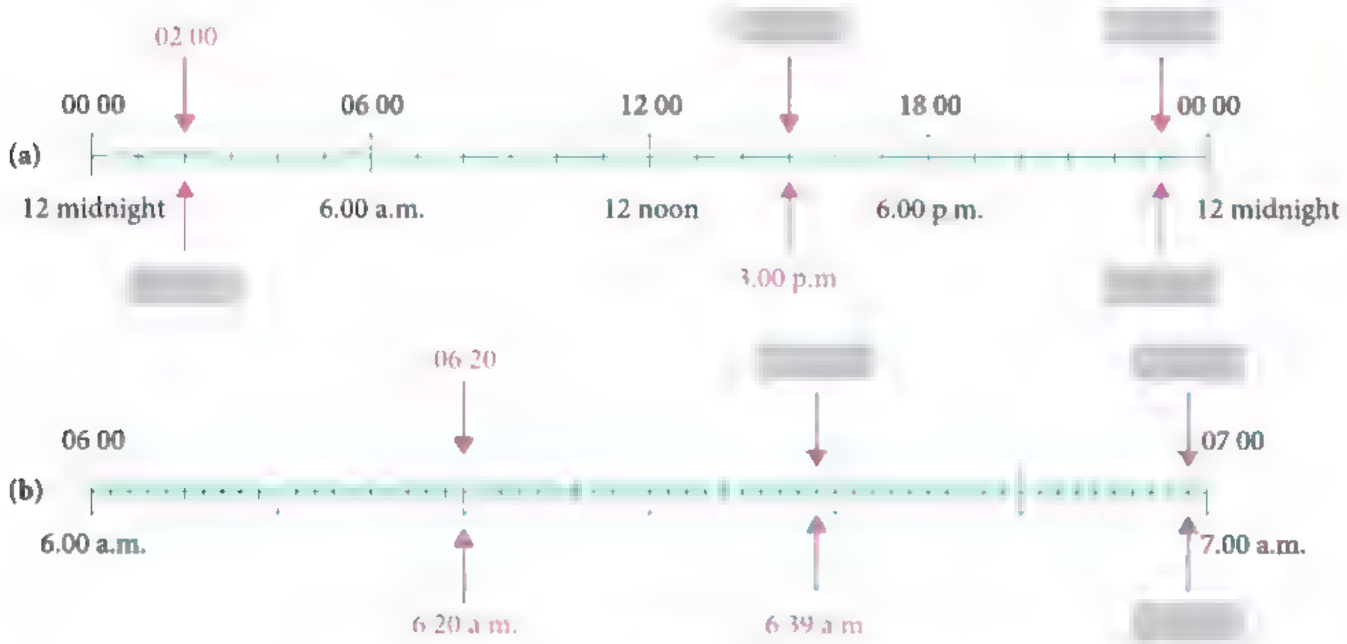


Fig. 9.2

The intervals in Fig. 9.2(**b**) represent the minutes from 6 a.m. to 7 a.m. How many minutes are there in an hour?

Practise Now 14A

1. Express each of the following in the 24-hour format.
   (a) 9.15 a.m. **(b)** 8.59 p.m. **(c)** 12.10 a.m.
2. Express each of the following in the 12-hour format.
   (a) 00 08 **(b)** 02 10 **(c)** 12 56
3. What is the time one minute after 23 59 in the 24-hour format?

Worked Example 14

Finding duration between two given times

Cheryl left her house at 09 10 for Town P. If she reached Town P at 12 06, how long did she take to travel to Town P?

Solution

∴ Cheryl took 2 hours and 56 minutes to travel to Town P.

Problem-solving Tip

Since 12 06 is close to 12 10 which is 3 hours from 09 10, we can use this as a reference time. 12 06 is 4 minutes before 12 10. Hence, the duration of Cheryl's journey is 4 minutes less than 3 hours, i.e. 2 hours 56 minutes.

1. Nadia left her house at 6.10 a.m. for her morning run, and returned at 7.33 a.m.. What is the duration of her run?
2. Albert and his family took a bus at 06 45 to City S. Immediately upon arrival at City S, they took a transit train to the city centre. If the train ride took 35 minutes and they arrived at the city centre at 11 23, determine
   (i) the time at which they arrived at City S,
   (ii) the duration of the bus ride to City S.

Exercise 9C

Time zones

The following shows the flight itinerary for a return flight between Pakistan and Singapore

YOUR TICKET-ITINERARY

Flight	From		To		Status
SS 123	Lahore Tuesday 9 May 2023	01 50	Singapore Tuesday 9 May 2023	12 55	Confirmed
SS 124	Singapore Sunday 14 May 2023	12 20	Lahore Sunday 14 May 2023	17 35	Confirmed

Fig. 9.3

1. Given that the flight duration from Pakistan to Singapore is 8 h 5 min, determine the arrival time in Singapore.
2. Explain why there is a difference between your answer to Question 1 and the arrival time stated in Fig. 9.3.
3. Determine the time difference between the time zones in Pakistan and in Singapore. State if the local time in Singapore is ahead or behind the local time in Pakistan.
4. Based on your answer in Question 3, calculate the flight duration from Singapore to Pakistan.

Information

The Earth completes a rotation of 360° about its axis over 24 hours. Every hour, it moves about 15°. Time zones (each 15° of longitude wide) were introduced to ensure the standardisation of day and night across the planet, whereby the sun would be near its highest point in the sky at noon. The Earth is divided into 24 time zones, each 1 hour apart from its neighbouring section.
The time in each time zone is defined using a time standard known as the Coordinated Universal Time (UTC). With this reference, Pakistan is located in UTC+05:00, five hours ahead of UTC, while Singapore is located in UTC+08:00. Some countries, such as the United States, use more than one time zone. How many time zones does Pakistan use?

In Worked Example 14, we determined the duration of Cheryl's journey by simply finding the difference between her departure time from her house and arrival time at Town P. However, as we see from the above Class Discussion, when a journey involves different countries, it is important to consider the difference in local times of each country when determining arrival or departure times and duration.

A direct flight from Tokyo (Japan) to Bangkok (Thailand) takes 6 hours and 50 minutes. A plane departs Tokyo at 8.45 p.m. on Sunday, 6 August 2023. If Japan is 2 hours ahead of Thailand, determine the date and time at which the plane arrives in Bangkok.

There are 60 minutes in an hour. Likewise, there are 60 seconds in a minute. How do we convert between these units of measurement?

Worked Example 15

Converting units of time

Express

(a) 45 min in hours as a fraction, (b) 3060 s in hours as a decimal,

(c) $\frac{1}{3}$ min in seconds, (d) 0.64 h in seconds.

Solution

(a) $60 \text{ min} = 1 \text{ h}$

$1 \text{ min} = \frac{1}{60} \text{ h}$

$45 \text{ min} = 45 \times \frac{1}{60} \text{ h}$

$= \frac{45}{60} \text{ h}$

$= \frac{3}{4} \text{ h}$ reduce to the simplest form

(b) Method 1:

$60 \text{ s} = 1 \text{ min}$

$1 \text{ s} = \frac{1}{60} \text{ min}$

$3060 \text{ s} = 3060 \text{ s} \times \frac{1}{60} \text{ min}$

$= 51 \text{ min}$

$60 \text{ min} = 1 \text{ h}$

$1 \text{ min} = \frac{1}{60} \text{ h}$

$51 \text{ min} = 51 \times \frac{1}{60} \text{ h}$

$= \frac{51}{60} \text{ h}$

$= \frac{85}{100} \text{ h}$ express denominator as a power of ten

$= 0.85 \text{ h}$

Method 2:

$1 \text{ h} = 60 \text{ min}$

$= 60 \times 60 \text{ s}$

$= 3600 \text{ s}$

$1 \text{ s} = \frac{1}{3600} \text{ h}$

$3060 \text{ s} = 3060 \text{ s} \times \frac{1}{3600} \text{ h}$

$= \frac{3060}{3600} \text{ h}$

$= 0.85 \text{ h}$

(c) $1 \text{ min} = 60 \text{ s}$

$$\frac{1}{3} \text{ min} = \frac{1}{3} \times 60 \text{ s}$$
$$= 20 \text{ s}$$

(d) $1 \text{ h} = 3600 \text{ s}$

$$0.64 \text{ h} = 0.64 \times 3600 \text{ s}$$
$$= 2304 \text{ s}$$

Reflection

Part (a): We expressed 1 min = $\frac{1}{60}$ h. Would you have expressed 1 minute in hours as a decimal? Why or why not?
Part (b): How is Method 2 similar to Method 1? Which method do you prefer? Why?

1. Express
   (a) 18 min in hours as a fraction,
   (b) 4 h 12 min in hours as a decimal,
   (c) 855 s in hours as a decimal.

2. Express each of the following in the stated unit.
   (a) 2 h 9 min in minutes
   (b) $4\frac{2}{5}$ min in minutes and seconds
   (c) 0.45 h in minutes
   (d) $\frac{7}{20}$ h in seconds

B. Concept of speed (Recap)

In primary school, we have learnt that the **speed** of an object is the distance travelled by the object per unit time:

$$\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

The speed of an object tells us how fast it is moving. Speed can be expressed in different units, such as m/s, km/h, m/min and cm/s.

Measures
Speed is a measure used to analyse and compare how fast objects are moving. It is obtained by dividing the measure of distance by the measure of time.
In other words, we can sometimes combine the measures of two attributes to derive a new measure.

16

Problem involving speed

A ferry travels 20 km from the Point A to Point B. If it starts its journey from Point A at 10.50 a.m. and reaches Point B at 11.30 a.m. on the same day, calculate the speed of the ferry in

(i) km/h, (ii) m/s.

Solution

(i) Duration from 10.50 a.m. to 11.30 a.m. = 40 min

$$= \frac{40}{60}\text{ h}$$

$$= \frac{2}{3}\text{ h}$$

$$\text{Speed of ferry} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{20\text{ km}}{\frac{2}{3}\text{ h}}$$

$$= 20 \div \frac{2}{3}\text{ km/h}$$

$$= {}^{10}\cancel{20} \times \frac{3}{\cancel{2}_1}\text{ km/h}$$

$$= 30\text{ km/h}$$

1 km = 1000 m
1 h = 60 min
= 60 × 60 s
= 3600 s

(ii) $30\text{ km/h} = \frac{30\text{ km}}{1\text{ h}}$

$$= \frac{30 \times 1000\text{ m}}{3600\text{ s}}$$ convert 30 km into m and 1 h into s

$$= \frac{30\ 000\text{ m}}{3600\text{ s}}$$

$$= \frac{25}{3}\text{ m/s}$$

$$= 8\frac{1}{3}\text{ m/s}$$

Problem-solving Tip

(ii) If the question asks for the speed of the ferry in m/s without first asking for it in km/h, you can convert 20 km to 20 000 m, and 40 min to 2400 s, before finding its speed in m/s.

Practise Now 16

Exercise 9C

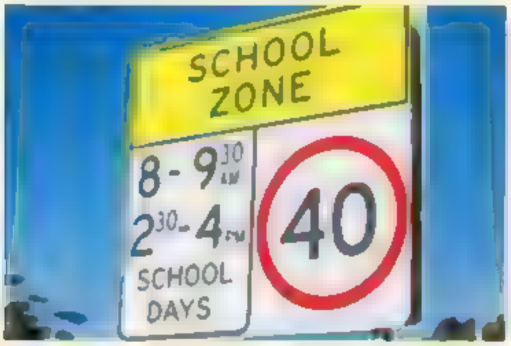
1. A train travels 16.8 km in 25 minutes. Find the speed of the train in
   (i) km/h, (ii) m/s.
2. A car travels at a speed of 55 km/h. Find the distance travelled by the car in 12 minutes and 30 seconds, giving your answer in metres.
3. In Pakistan, the speed limits for cars depend on the type of roads. The highest speed limit for cars is 120 km/h on motorways. Express this speed in
   (i) m/s, (ii) cm/min.

Reflection

For Question 3, which unit of the speed limit is more meaningful? Why?

4. The sailfish is the fastest marine animal, with a fastest recorded leap of 109 km/h. In August 2022, the world record for the men's 100 m freestyle was set at 46.86 seconds. How many times faster is the sailfish than this swimmer?

1. The picture shows the speed limit in a school zone in Australia.



When Ali drove along this road during the stated hours on a school day, he was travelling at 30 m/s. Should he continue driving at that speed?

2. The greatest human running speed on record is 12.4 m/s.
The record is held by Usain Bolt who attained this speed in a 100 metre sprint in 2009. He took 9.58 seconds to complete this sprint.
   (a) **(i)** Calculate Usain Bolt's average speed during the sprint.
   (ii) Explain why your answer in (i) is not 12.4 m/s.
   (b) The maximum speed of a zebra on record is 40 miles per hour.
   Given that 1 mile is approximately 1.6 km, determine whether this zebra is faster than Usain Bolt.

C. Constant speeds and average speeds

If the speed of an object does not change throughout its journey, the object is said to be travelling at a **constant speed**.
However, in real-life situations, it is unlikely for an object to travel at the same speed throughout its journey. Why?
The ferry's speed of 30 km/h in Worked Example 16 is actually its **average speed**.
This means that on average, the ferry travels 30 km every hour.

Information

A ferry, like a car, is equipped with a speedometer, which gives its speed at a particular instant. The reading of the speedometer will change from time to time because its speed is not constant.

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Worked Example 17

Problem involving average speed

Nadia sets off from her home on a 70-km journey to her friend's house. She travels the first 40 km of her journey at a speed of 40 km/h and the remaining at a speed of 60 km/h. Calculate the average speed for her entire journey.

$$\text{Time taken for first part of journey} = \frac{\text{distance travelled}}{\text{average speed}}$$
$$= \frac{40}{40}$$
$$= 1 \text{ h}$$

$$\text{Time taken for second part of journey} = \frac{\text{distance travelled}}{\text{average speed}}$$
$$= \frac{70-40}{60}$$
$$= \frac{1}{2} \text{ h}$$

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$
$$= \frac{70}{1+\frac{1}{2}}$$
$$= 46\frac{2}{3} \text{ km/h}$$

In a triathlon, Raju swims a distance of 1.5 km at an average speed of 2.5 km/h, cycles 40 km in $1\frac{1}{2}$ hours and runs at an average speed of 9 km/h for $1\frac{1}{9}$ hours. Find his average speed for the entire competition.

1. In Worked Example 17, Shaha did this calculation:

$$\text{Average speed} = \frac{\text{speed for } 1^{\text{st}} \text{ part of journey} + \text{speed for } 2^{\text{nd}} \text{ part of journey}}{2}$$
$$= \frac{40+60}{2}$$
$$= 50 \text{ km/h}$$

How would you explain to Shaha that she is wrong?

2. In Chapter 8, we have learnt that we should not add or take the average of two or more percentages. Compare this with Question 1. Why is it that we should not add or take the average of two or more speeds?

From the Thinking Time on page 234, we learn that:

In general, we should [illegible] add speeds or take average of speeds because the [illegible] may be different.

For speeds, the bases refer to the times taken for different parts of the journey.

Worked Example 18

Average speed problem involving algebra

A car travels at an average speed of 60 km/h from Town A to Town B. If the car travels at an average speed of 72 km/h instead, it would reach Town B 15 minutes earlier. Calculate the distance between Town A and Town B.

Solution

Let the distance between Town A and Town B be x km.

Then time taken to travel from Town A to Town B at an average speed of 60 km/h = $\frac{x}{60}$ h,

and time taken to travel from Town A to Town B at an average speed of 72 km/h = $\frac{x}{72}$ h.

$$\therefore \frac{x}{60} - \frac{x}{72} = \frac{15}{60}$$

$$\frac{x}{360} = \frac{1}{4}$$

$$x = 90$$

Distance between Town A and Town B = 90 km

Reflection

$\frac{x}{60} - \frac{x}{72} = \frac{15}{60}$ is the same as

$\frac{x}{72} + \frac{15}{60} = \frac{x}{60}$

In other words, we are adding durations here.

If we should not add speeds in general, why can we always add durations and distances?

Practice Now 18

Similar and Further Questions

Exercise 9C

A car leaves Town A for Town B, which are 550 km apart, at an average speed of 72 km/h. At the same time, a truck leaves Town B for Town A and travels along the same road as the car at an average speed of 38 km/h. Find the time taken for the two vehicles to meet.

1. What is the difference between
   (a) rate and speed?
   (b) constant speed and average speed?
2. What have I learnt in this section or chapter that I am still unclear of?

Basic | Intermediate

Exercise 9C

1. Find the duration between
   (a) 4.15 p.m. and 9.29 a.m.
   (b) 12 16 and 20 58
   (c) 04 35 and 09 32
   (d) 13 54 and 2.01 p.m.

2. Waseem spent 3 h 15 min at a fun fair. If he arrived at 14 20, what time did Waseem leave the fun fair?

3. A flight which left Hong Kong at 5.35 p.m. arrived at Tokyo on the same day at 10.50 p.m. If Tokyo is an hour ahead of Hong Kong, how long is the flight duration?

4. Express
   (a) 49 s in minutes as a fraction,
   (b) 36 min in hours as a decimal,
   (c) 66 s in minutes as a decimal,
   (d) 2 min 54 s in minutes as a decimal,
   (e) 945 s in hours as a decimal,
   (f) 25 min 3 s in hours as a fraction.

5. Express each of the following in the stated units.
   (a) 0.13 min to seconds
   (b) $\frac{1}{3}$ h in minutes
   (c) $1\frac{5}{6}$ h in seconds
   (d) 0.96 h in seconds

6. A particle travels 24.6 km in 30 minutes. Find the speed of the particle in
   (i) km/h, (ii) m/s.

7. A high-speed train travels at a speed of 200 km/h. If the train sets off from Station A at 1224 hours, and reaches Station B at 1412 hours, find the distance between the two stations, giving your answer in metres.

8. Express each of the following in km/h.
   (a) 8.4 km/min (b) 315 m/s
   (c) 242 m/min (d) 125 cm/s

9. Express each of the following in m/s.
   (a) 65 cm/s (b) 367 km/h
   (c) 1000 cm/min (d) 86 km/min

10. A bullet train travels at a speed of 365 km/h. In July 2023, Pakistan's top sprinter Shajar Abbas set a new national record for the men's 100 m sprint at 10.37 seconds. How many times is a bullet train as fast as the fastest Pakistani sprinter?

11. A car travels the first 19 km of its journey at an average speed of 57 km/h and the remaining 55 km at an average speed of 110 km/h. Find the average speed of the car for its entire journey.

12. Cheryl arrived at the theatre 15 minutes before the musical started. The musical was 2 hours 12 minutes long, excluding a 20-minute intermission. Given that the musical ended at 10.06 p.m., what time did Cheryl arrive at the theatre?

13. Melbourne is 2 hours ahead of Singapore. A flight from Melbourne to Singapore scheduled to depart at 14 05 was delayed by 1 hour 20 minutes. Find the arrival time in Singapore if the flight duration was 7 hours 35 minutes.

14. A car and a bus are travelling towards each other. They are 510 km apart at 1320 hours and they pass each other at 1620 hours. If the car is travelling at a speed of 90 km/h, find the speed of the bus.

Exercise

15. Two points, X and Y, are 120 m apart. M is the midpoint of X and Y. An object travels from X to M in 12 seconds and then from M to Y at an average speed of 15 m/s. Find

(i) the time taken for the object to travel from M to Y,

(ii) the average speed of the object for its entire journey from X to Y.

16. Two points, L and N, are 160 m apart. M lies on the straight line joining L and N. An object travels from L to M at an average speed of 10 m/s in 6 seconds and then from M to N at an average speed of 25 m/s. Find the average speed of the object for its entire journey from L to N.

17. A car travels the first 50 km of its journey at an average speed of 25 m/s and the next 120 km at an average speed of 80 km/h. The car completes the last part of its journey at an average speed of 90 km/h in 35 minutes. Find the average speed for its entire journey, giving your answer in km/h.

18. OPEN A car travels at a speed of x km/h for 4 hours and at a speed of y km/h for the next 2 hours. Its average speed for the entire journey is 60 km/h. Suggest two possible values of x and y.

19. A passenger train travels at a speed of 72 km/h. A man on the passenger train observes a goods train travelling at a speed of 54 km/h in the opposite direction. If the goods train passes him in 8 seconds, find the length of the goods train.

20. David leaves Town A and travels towards Town B at an average speed of 100 m/min. At the same time, Ken and Li Ting travel from Town B towards Town A at an average speed of 80 m/min and 75 m/min respectively. If David meets Li Ting 6 minutes after passing Ken, find the distance between Town A and Town B.

21. On his outward journey, Ali travelled at a speed of s km/h for $2\frac{1}{2}$ hours. On his return journey, he increased his speed by 4 km/h and saved 15 minutes. Find Ali's average speed for the whole journey.

In this chapter, we learnt how ratio can be used to compare two or more numbers, or quantities of the same kind. By extending the concept of ratio to include how we can compare quantities of different kinds, we can develop another measure, known as rate, to quantify new physical attributes. For example, speed is a rate that compares distance with time. Rate is a powerful concept with many real-world applications such as exchange rates, fuel consumption rates, interest rates, population growth rates, rates of infection, death rates, birth rates, and even rates of spreading of rumours! If two quantities are directly proportional to each other, the rate of change of one quantity with respect to the other quantity is a constant.

Here, we see that a single, simple, yet powerful idea such as ratio or rate can be used to compare physical attributes in the real world. However, it is not easy to measure or compare subjective attributes such as happiness, poverty, popularity, using some form of rate or ratio. Are there any attributes that you would like to measure or compare? Can you use a rate or a ratio?

1. **Ratio**
   (a) A ratio is a way of comparing two or more quantities of the *same kind* that either have no units (i.e. just numbers) or are measured in the same unit.
   (b) The ratio $a : b$, where a and b are positive numbers, has *no units*.
   (c) The ratio $a : b$, where a and b are positive numbers, can be expressed in an equivalent form as a *fraction* $\frac{a}{b}$.
   - Give three examples of how ratios are used in a real-world context.

2. **Rate**
   (a) A rate is a way of comparing how one quantity *changes* with another quantity. The two quantities can be of the same kind or of different kinds.
   (b) A rate can be expressed as the number of units of one quantity *per n units* of another quantity, where n is a positive integer, or it can be expressed in terms of categories or brackets depending on the context.
   (c) A rate that compares two quantities of the same kind can be converted to a ratio and vice versa.
   - Give an example of a rate that can be converted to a ratio.

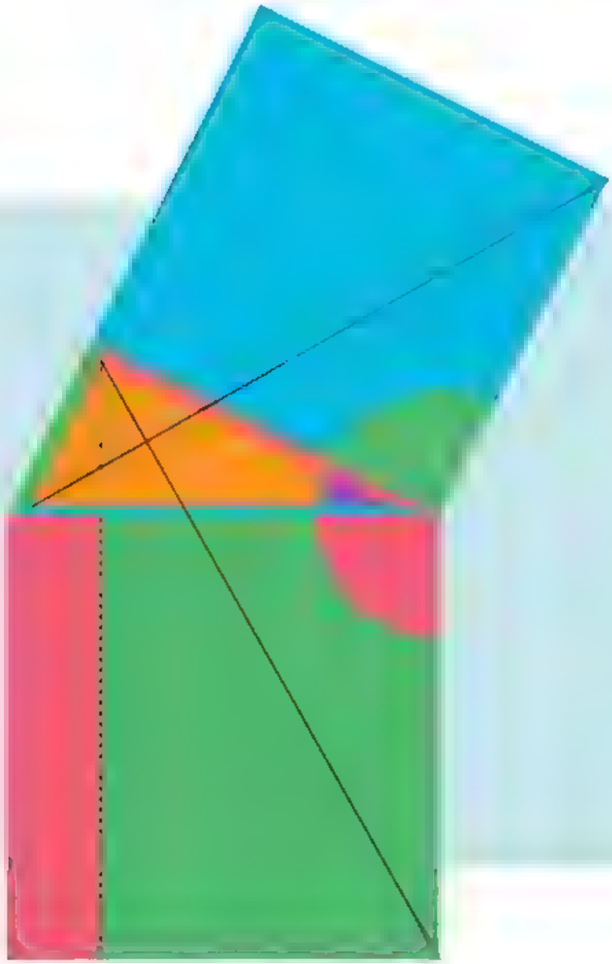
3. **Speed**
   (a) An object is said to be travelling at a **constant speed** when its speed does not change throughout the journey.
   (b) The formula for the **average speed** of an object is:

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

   - Think of a real-life problem that you can solve with the help of the above formula.

CHAPTER 10

Basic Geometry



Geometry is one of the earliest branches of mathematics developed by mankind. A series of books written by Euclid of Alexandria called 'The Elements' at around 300 BC has characterised much of geometry till today. In this chapter, we are going to examine the properties of familiar geometrical objects, like lines and angles, to see how they can be used to model real-world situations.

Learning Outcomes

What will we learn in this chapter?

- What complementary and supplementary angles are
- How to identify various types of angles (i.e. acute, right, obtuse, straight and reflex angles)
- How to use properties of angles formed by intersecting lines (i.e. adjacent angles on a straight line, angles at a point and vertically opposite angles) to solve geometrical problems
- How to use properties of angles formed by two parallel lines and a transversal (i.e. corresponding angles, alternate angles, interior angles and their converse) to solve geometrical problems
- Why angle properties have useful applications in real life

Introductory Problem

Some carparks are designed as angled parking lots (as shown in Fig. 10.1). These carparks use less space, as less room is required for a car to drive in and out of the lot.

The outlines of each parking lot have to be parallel to each other. How do painters ensure that the lines are painted accurately?

Fig. 10.1

10.1 Basic geometrical concepts and notation

In **geometry**, we study shapes, relative positions of objects and properties of space.

A. Points, lines and planes

In primary school, we have learnt about points and lines. Other mathematical (or geometrical) objects we will learn are line segments, rays and planes, as listed in Table 10.1.

Object	Description	Pictorial representation
Point	• Most basic geometrical object • A collection of points make up other geometrical objects. • A point has *zero dimension* (no size). • We use a capital letter to label or name a point, e.g. point A or point B.	We mark a point with a cross or a dot. A *cross* (see Point A below) is generally preferred as the size of a dot affects its accuracy. × • Point A Point B
Line segment	• A line segment AB is formed by joining two points A and B. • We label or name a line segment by its endpoints, e.g. line segment AB. • A line segment is made up of an infinite number of points between its two endpoints.	A line segment ends directly at the two endpoints. A B
Line	• A line l is formed if we extend a line segment AB indefinitely. • We use a small letter to label a line, e.g. line l. • A line has *one dimension*: it has an indefinite length but no breadth or thickness.	A line has an indefinite length and extends beyond the two points A and B, unlike a line segment. l A B
Ray	• A ray is a line with only one endpoint. • An example of a ray is light from a source such as the Sun or a lamp.	An arrow at one end indicates that the ray extends indefinitely in that direction. H

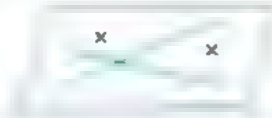
Object	Description	Pictorial representation
Plane	• A plane is a flat *two-dimensional* surface: it has a length and a breadth, but no thickness. • It is made up of an infinite number of points. • The floor is an example of a horizontal plane and a wall is an example of a vertical plane.	A slanted rectangle can be used to represent a plane. When there is no need to draw one, the paper on which the dots and lines are drawn is the plane. 

Table 10.1

Are the following statements true or false?
If it is false, give the correct statement.

(a) The two endpoints are the only two points of a line segment.
(b) There is exactly one line that passes through any two distinct (i.e. different) points on a plane.
(c) It is not possible to draw a line that passes through three distinct points on a plane.
(d) Any two lines on a plane will intersect at one point.
(e) If two points lie on a plane, then the line that passes through the two points lies on the same plane.

From the above Thinking Time, we have observed that:

- If a line can be drawn through three distinct points on a plane as shown in Fig. 10.2**(a)**, then the three points are **collinear**.
- If two lines on a plane intersect at X as shown in Fig. 10.2**(b)**, they are **intersecting lines**. X is the *point of intersection*.
- If two intersecting lines intersect each other at right angles as shown in Fig. 10.2**(c)**, they are **perpendicular lines**, i.e. $AB \perp PQ$ (or $AB \perp PQ$). F is the *foot of the perpendicular* from P to AB. We draw a right angle ⊾ to indicate that the lines are perpendicular to each other.
- If two lines on a plane do not intersect at any point as shown in Fig. 10.2**(d)**, they are **parallel lines**, i.e. $AB \;//\; CD$. A pair of arrowheads pointing in the same direction indicate that the lines are parallel to each other.

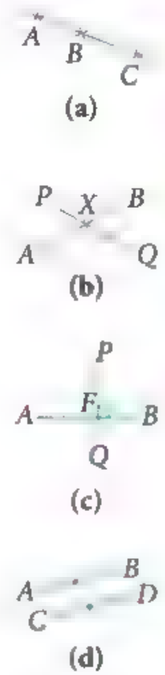


Fig. 10.2

The notations ⊥ and // help to convey the idea that the lines are perpendicular and parallel respectively, in a concise and precise manner.
What other notations and conventions do we use in this chapter?

Geometrical diagrams often contain information (or features) such as equal angles or sides, perpendicular or parallel lines, which can help us visualise relationships between points, lines and angles when solving problems related to geometrical figures or the real-world objects that they model.

Although a line and a surface can also be curved, we will only be dealing with *straight lines* and planes in this chapter. Note that a plane is always flat, unlike a curved surface.

B. Angles

An **angle** is formed when two rays OA and OB share the same endpoint O.

Point O is the **vertex** of the angle while OA and OB are the **sides** (or arms) of the angle.

The angle is called **angle** AOB or angle BOA and is written as $\angle AOB$ or $\angle BOA$. Another way of writing this angle is $A\hat{O}B$ or $B\hat{O}A$.

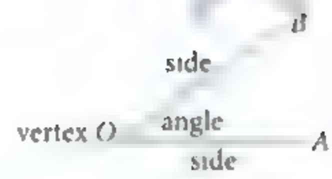


Fig. 10.3

We can also label the angle using a small letter such as x and write the angle x as $\angle x$.

Recall that if we want know how long a line segment is, we measure its length.

If we want to know how large an angle is, we measure its [illegible]. The angle (measure) is the amount of rotation from one of the sides to the other.

A [illegible] is a way of quantifying a property of an object. For example, length measures the distance between two points and angle measures the amount of rotation. These measurements have units, e.g. 2 cm, 40°. One *purpose* of a measurement is to analyse compare and order the property of two or more objects.

The standard unit for measuring angles is the **degree** (°). It is defined as $\frac{1}{360}$ of a complete revolution. Thus, one complete revolution about a point is 360°.

Fig. 10.4

Angles are measured using a **protractor**.

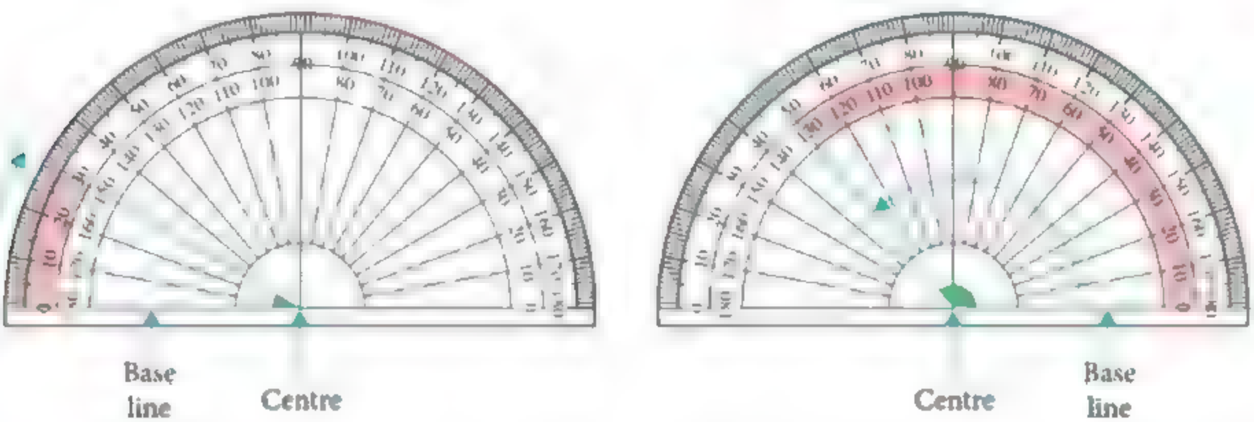


Fig. 10.5

[illegible] Place a protractor such that its centre is at the vertex of the angle and its base line is along one side of the angle.

[illegible] Read off the angle from the *outer* scale in Fig. 10.5(**a**). The angle is

[illegible] Read off the angle from the *inner* scale in Fig. 10.5(**b**). The angle is

C. Types of angles

In primary school, we have learnt about acute, right and obtuse angles. We will now learn two more: straight angle and reflex angle.

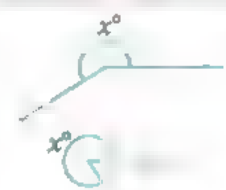
Name	Definition	Illustration	Real-life example
Acute angle	$0° < x° < 90°$		
Right angle	$x° = 90°$		
Obtuse angle	$90° < x° < 180°$		
Straight angle	$x° = 180°$		
Reflex angle	$180° < x° < 360°$		

Table 10.2

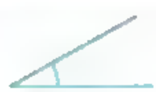
Attention

- We use $x°$ for the value of the angle, i.e. x is a *number without any unit*.
- The names of the angles from the smallest to the largest size (excluding right angle and straight angle) are in alphabetical order: acute angle, obtuse angle, reflex angle. This may help you remember the names better.

Just For Fun

A magnifying glass can enlarge an object to three times its original size.
How many degrees will an angle of 2° appear to a man using the magnifying glass?

Classify each of the following as an acute, right, obtuse, straight or reflex angle.

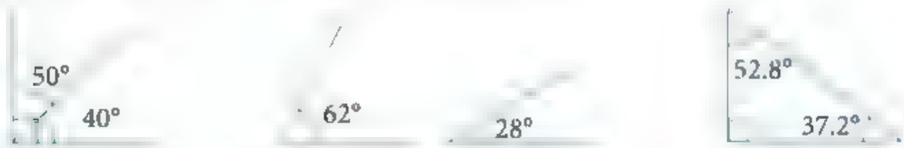
(a) 
(b) 
(c)

(d) 
(e) 
(f) 

(g) 176°
(h) 326°
(i) 48°

D. Complementary and supplementary angles

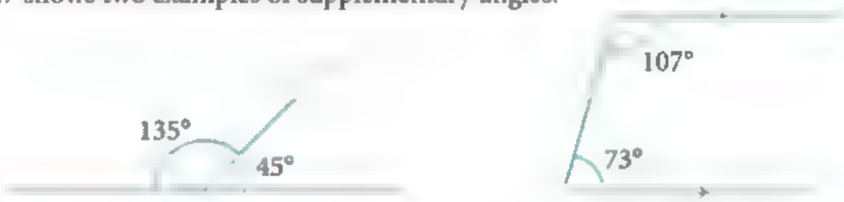
Two angles are **complementary** when they add up to 90°.
Fig. 10.6 shows examples of complementary angles.



(a) 40° and 50° (b) 62° and 28° (c) 37.2° and 52.8°

Fig. 10.6

Two angles are **supplementary** when they add up to 180°.
Fig. 10.7 shows two examples of supplementary angles.



(a) (b)

Fig. 10.7

Attention

'Complementary angles' and 'supplementary angles' are terminologies that describe only *two angles* that add up to 90° or 180°. They are *not* angle properties.

10.2 Properties of angles formed by intersecting lines

A. Adjacent angles on a straight line

Fig. 10.8 shows two examples of adjacent angles on a straight line. *Adjacent angles* are angles that

- share a common vertex,
- have a common side,
- lie on opposite sides of the common side.

(a) (b)

Fig. 10.8

Attention

$\angle a$ and $\angle b$ are *not* adjacent angles in the following examples

- 

no common side

- 

do not lie on opposite sides of common side

Adjacent angles on a straight line

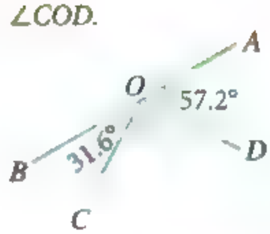
The sum of adjacent angles on a straight line is 180°.
(Abbreviation: adj. ∠s on a str. line)

Adjacent angles on a straight line

Given that AOB is a straight line, calculate

(a) the value of a,

(b) $\angle COD$.



Solution

(a) $a° + 49° = 180°$ (adj. ∠s on a str. line)

$a° = 180° - 49°$

$= 131°$

$\therefore a = 131$

(b) $57.2° + \angle COD + 31.6° = 180°$ (adj. ∠s on a str. line)

$\angle COD = 180° - 57.2° - 31.6°$

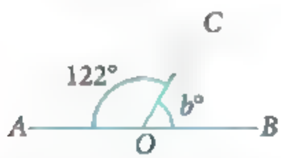
$= 91.2°$

Attention

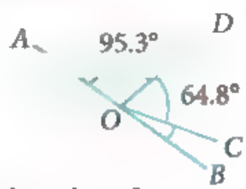
- State the angle property that you have used in your working.
- The degree symbol (°) is included as a unit when finding an angle, such as $\angle COD$ in **(b)**. The unknown a in **(a)** has no unit.

1. Given that AOB is a straight line, calculate

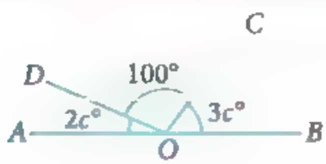
(a) the value of b,



(b) $\angle COB$.



2. In the figure, AOB is a straight line. Find the value of c.



B. Angles at a point

Fig. 10.9 shows an example of angles at a point.

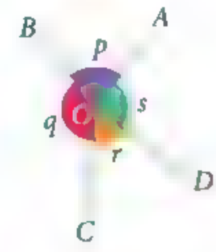


Fig. 10.9

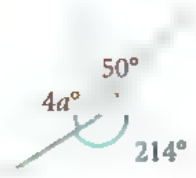
Angles at a point

The sum of angles at a point is 360°.
(Abbreviation: ∠s at a point)

Worked Example 2

Angles at a point

Calculate the value of a in the figure.

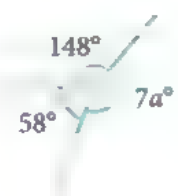


Solution

$$4a° + 50° + 214° = 360° \text{ (∠s at a point)}$$
$$4a° = 360° - 50° - 214°$$
$$= 96°$$
$$a° = 24°$$
$$\therefore a = 24$$

Exercise 10A

1. Find the value of a in the figure.



2. In the figure, POQ is a straight line. Find the value of b.

Reflection

Is there more than one way to solve Question 2? Which way do you prefer?

C. Vertically opposite angles

In Fig. 10.10, two straight lines AB and CD intersect at the point O. $A\hat{O}C$ and $B\hat{O}D$ are called *vertically opposite angles*. $B\hat{O}C$ and $A\hat{O}D$ are also vertically opposite angles.

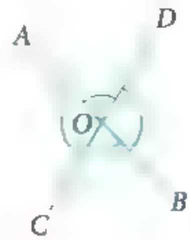


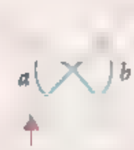
Fig. 10.10

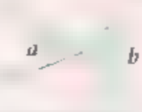
Vertically opposite angles

Vertically opposite angles are
(Abbreviation: vert. opp. ∠s)

Attention

The following shows some examples of angles that are not vertically opposite.

- 

- 

Vertically opposite angles

In the diagram, AOB and COD are straight lines. Copy to complete the following.

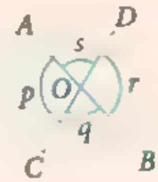
$\angle p + \angle q = 180°$ (adj. $\angle$s on a str. line)

$\angle q + \angle r =$ ______

$\therefore \angle p + \angle q = \angle q + \angle r$

$\therefore \angle p$ ______

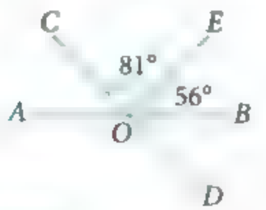
Similarly, $\angle q =$ ______.



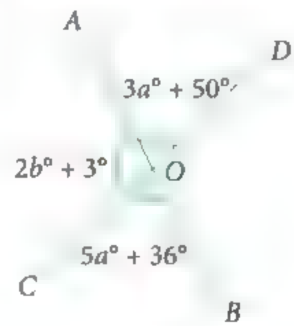
Worked Example

Vertically opposite angles

(a) In the figure, AOB and COD are straight lines. If $B\hat{O}E = 56°$ and $C\hat{O}E = 81°$, calculate $A\hat{O}D$.



(b) In the figure, AOB and COD are straight lines. Calculate the value of a and of b.



Solution

(a) $A\hat{O}D = 56° + 81°$ (vert. opp. $\angle$s)

$= 137°$

(b) $3a° + 50° = 5a° + 36°$ (vert. opp. $\angle$s)

$5a° - 3a° = 50° - 36°$

$2a° = 14°$

$a° = 7°$

$\therefore a = 7$

$5a° + 36° + 2b° + 3° = 180°$ (adj. $\angle$s on a str. line)

$5(7°) + 36° + 2b° + 3° = 180°$ — substitute $a = 7$

$2b° = 180° - 35° - 36° - 3°$

$= 106°$

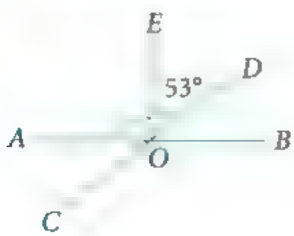
$b° = 53°$

$\therefore b = 53$

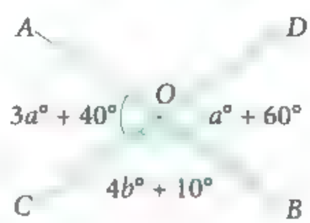
Problem-solving Tip

You can also find the value of b by considering the adjacent angles on the line COD.

(a) In the figure, AOB and COD are straight lines. If $A\hat{O}E = 90°$ and $D\hat{O}E = 53°$, calculate $B\hat{O}C$.

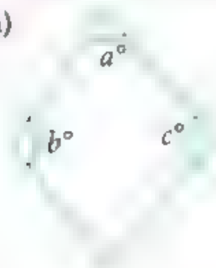


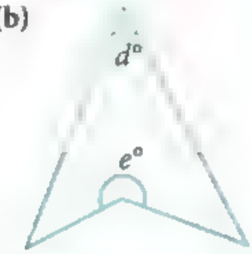
(b) In the figure, AOB and COD are straight lines. Calculate the value of a and of b.



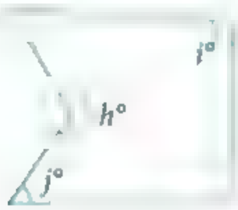
Exercise 10A

1. Using a protractor, measure and write down the value of each of the following unknowns.

(a)


(b)


(c)


(d)


2. Classify each of the following as an acute, right, obtuse, straight or reflex angle.

(a)

(b)


(c)


(d)


(e) 180°
(f) 181°
(g) 89°
(h) 124°

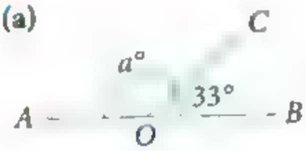
3. Find the complementary angle of each of the following angles.

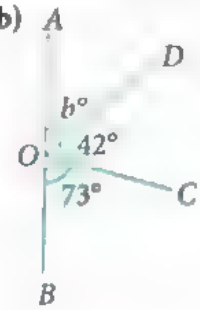
(a) 18°
(b) 46°
(c) 53°
(d) 64°

4. Find the supplementary angle of each of the following angles.

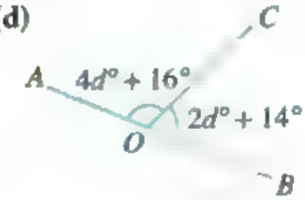
(a) 36°
(b) 12°
(c) 102°
(d) 171°

5. Given that AOB is a straight line, find the value of the unknown in each of the following figures.

(a)


(b)


(c)

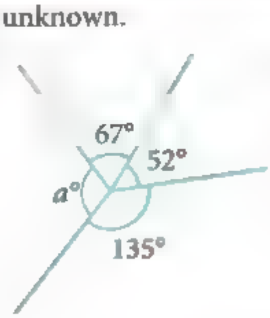

(d)


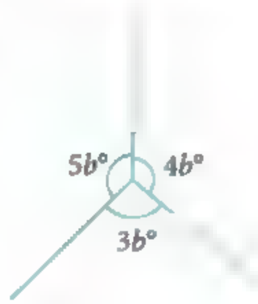
6. $x°$, $y°$ and $z°$ are three angles lying on a straight line.

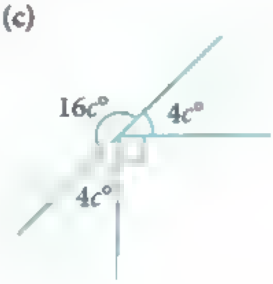
(a) If $y° = 45°$ and $z° = 86°$, find the value of x.

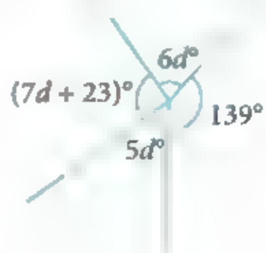
(b) If $x° = 2y°$ and $z° = 3y°$, find the value of y.

7. For each of the following figures, find the value of the unknown.

(a)


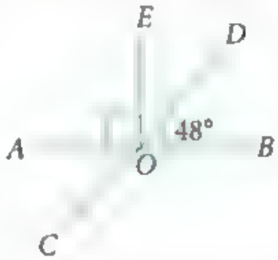
(b)


(c)


(d)


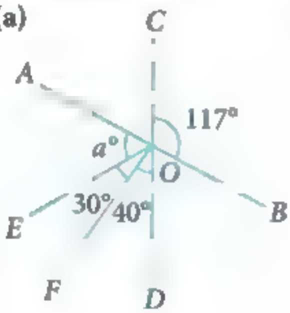
Exercise 10A

8. In the figure, AOB and COD are straight lines. If $A\hat{O}E = 90°$ and $B\hat{O}D = 48°$, find
(i) $A\hat{O}C$, (ii) $D\hat{O}E$.

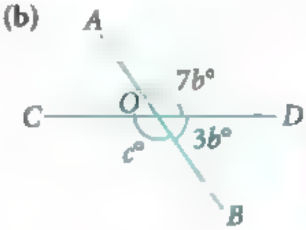


9. Given that AOB and COD are straight lines, find the value(s) of the unknown(s) in each of the following figures.

(a)

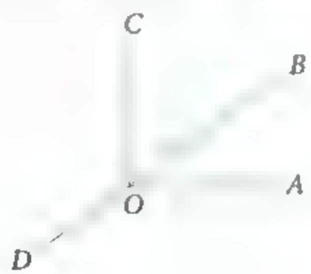


(b)



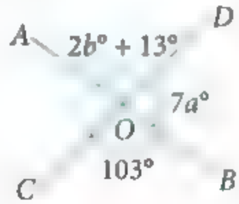
10. $x°$, $y°$ and $z°$ are three angles lying on a straight line.
(a) If $y° = x° + z°$, find the value of y.
(b) If $x° = y° = z°$, find the value of z.

11. In the figure, DOB is a straight line. If $B\hat{O}C$ is twice of $A\hat{O}B$, $C\hat{O}D$ is four times of $A\hat{O}B$ and $D\hat{O}A$ is five times of $A\hat{O}B$, find all the four angles.

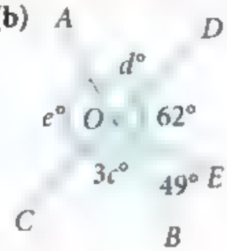


12. Given that AOB and COD are straight lines, find the values of the unknowns in each of the following figures.

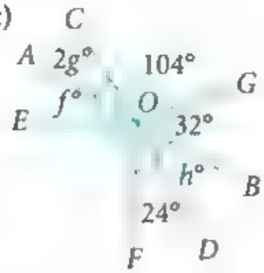
(a)



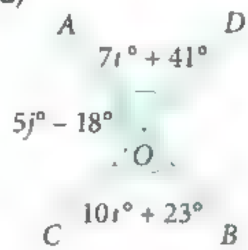
(b)



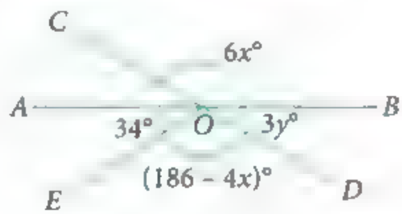
(c)



(d)



13. In the figure, AOB and COD are straight lines.
(i) Find the value of x and of y.
(ii) Find obtuse $A\hat{O}D$ and reflex $C\hat{O}E$.



10.3 Properties of angles formed by two parallel lines and transversal

A. Parallel lines and transversal

In Section 10.1A, we were introduced to *parallel lines*, which are lines on the same plane that do not intersect one another. They are represented by the same number of arrowheads pointing in the same direction as shown in Fig. 10.11.

We use the symbol '//' to denote 'is parallel to', e.g. in Fig. 10.11**(a)**, we write $AB // CD$, which means AB is parallel to CD.

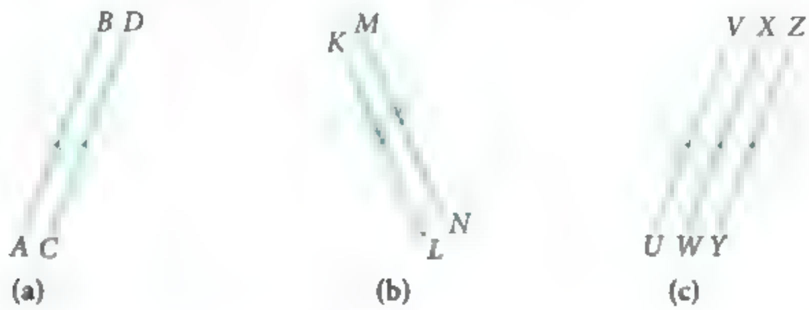


Fig. 10.11

Fig. 10.12 shows a pair of railway tracks which are parallel. What other parallel real-world objects can you think of?

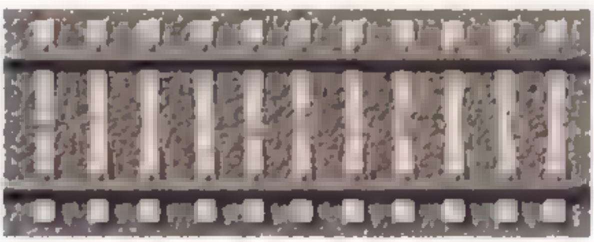


Fig. 10.12

A **transversal** is a line that crosses any two other lines, which may or may not be parallel. Fig. 10.13 shows two lines PQ and RS being cut by a transversal LM.

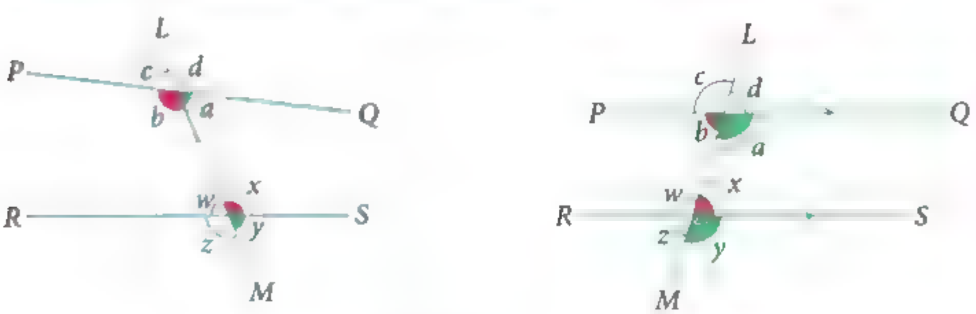
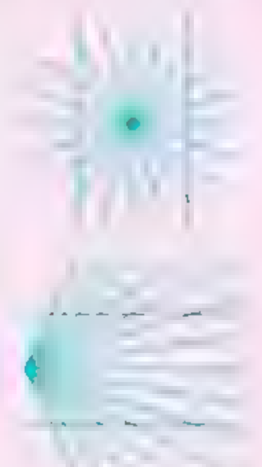


Fig. 10.13

- $\angle a$ and $\angle y$ are called *corresponding angles*. Name another pair of corresponding angles.
- $\angle b$ and $\angle x$ are called *alternate angles*. Name another pair of alternate angles.
- $\angle a$ and $\angle x$ are called *interior angles*. Name another pair of interior angles.

Just For Fun

Some parallel lines may not appear to be parallel. This can happen when there are other lines such as those shown in the following figures. Can you identify which lines are parallel?



B. Corresponding angles, alternate angles and interior angles

In this section, we will only look at the properties of angles formed by two *parallel* lines and a transversal.

Corresponding angles, alternate angles and interior angles

Go to www.sl-education.com/tmsoupp1/pg251 or scan the QR code on the right and open the geometry software template 'Parallel lines'. The template shows two *parallel* lines PQ and RS being cut by a transversal.

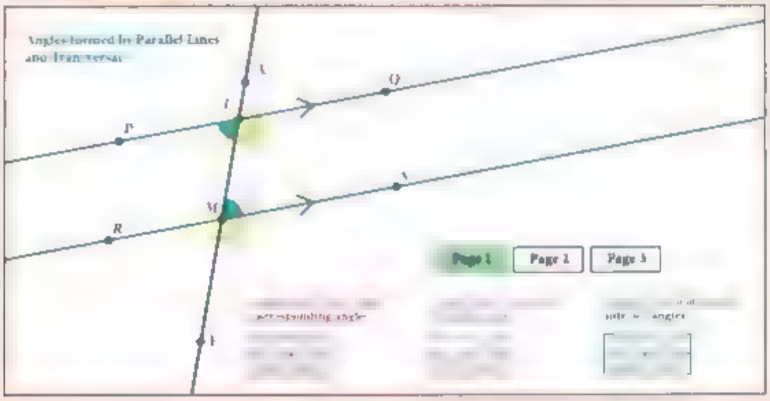


Fig. 10.14

1. $\angle a$ and $\angle b$ are corresponding angles. Change the sizes of $\angle a$ and $\angle b$ by clicking and dragging the points P, L, Q, R, M or S to move the parallel lines and the points X or Y to move the transversal. What is the relationship between $\angle a$ and $\angle b$?
2. $\angle c$ and $\angle d$ are alternate angles. Change the sizes of $\angle c$ and $\angle d$. What is the relationship between $\angle c$ and $\angle d$?
3. $\angle b$ and $\angle d$ are interior angles. Change the sizes of $\angle b$ and $\angle d$. What is the relationship between $\angle b$ and $\angle d$? Hint: Consider their sum.
4. Summarise the three angle properties that you have learnt from the above:
   (a) $\angle a = \angle$ ______ (corr. $\angle$s)
   (b) $\angle c = \angle$ ______ (alt. $\angle$s)
   (c) $\angle b + \angle d =$ ______ (int. $\angle$s)

Are the three angle properties also true if the lines PQ and RS are not parallel to each other?
Go to page 2 by clicking on the tab at the bottom right corner. Page 2 shows two *non-parallel* lines PQ and RS being cut by a transversal.

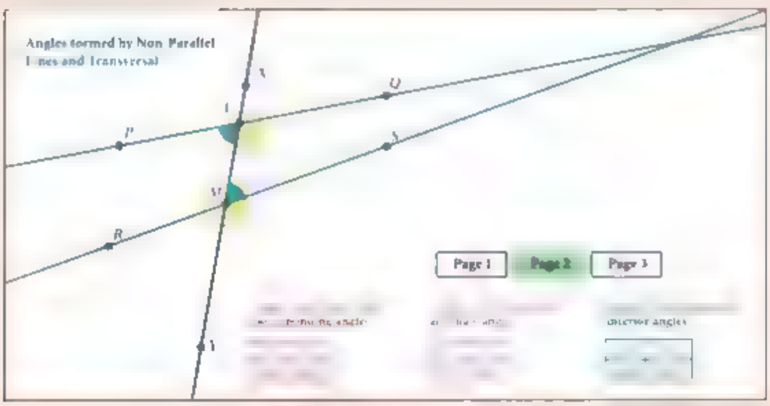


Fig. 10.15

5. Repeat the steps in Questions 1 to 3.
   Are the three angle properties in Question 4 also true for non-parallel lines PQ and RS?

But why are the three angle properties true for parallel lines PQ and RS?

Go to page 3, which shows two *parallel* lines PQ and RS being cut by a transversal.

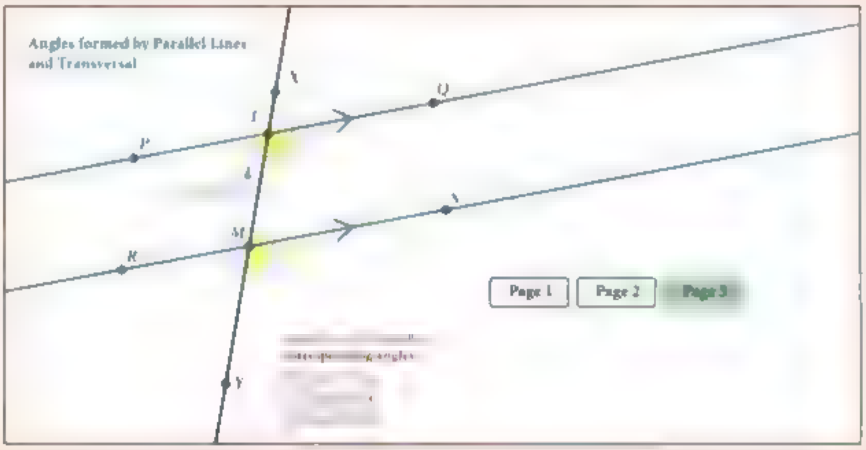


Fig. 10.16

6. $\angle a$ and $\angle b$ are corresponding angles. Click and drag the point M upwards towards the point L. This will translate $\angle a$ towards $\angle b$. Explain why $\angle b$ is equal to $\angle a$.

Go back to page 1 of the template.

7. $\angle c$ and $\angle d$ are alternate angles. How do you prove that they are equal?
   Use corresponding angles and vertically opposite angles.

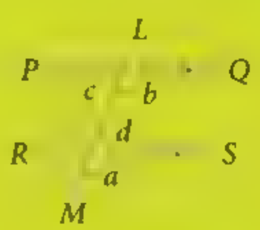
8. $\angle b$ and $\angle d$ are interior angles. Use two methods to prove that the sum of their angles is 180°.
   Use corresponding angles (or alternate angles) and adjacent angles on a straight line.

From the above Investigation, we have obtained the following three angle properties:

Angles formed by parallel lines and transversal

If two parallel lines, PQ and RS, are cut by a transversal LM, then

- corresponding angles are equal,
  e.g. $\angle a = \angle b$ (corr. $\angle$s, PQ // RS);
- alternate angles are equal,
  e.g. $\angle c = \angle d$ (alt. $\angle$s, PQ // RS);
- interior angles are supplementary,
  e.g. $\angle b + \angle d = 180°$ (int. $\angle$s, PQ // RS).



The *converse* for each of the above is also true, i.e. if two lines PQ and RS are cut by a transversal LM, and

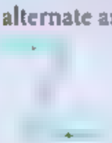
- if $\angle a = \angle b$, then PQ // RS (converse of corr. $\angle$s);
- if $\angle c = \angle d$, then PQ // RS (converse of alt. $\angle$s);
- if $\angle b + \angle d = 180°$, then PQ // RS (converse of int. $\angle$s).

Information

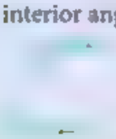
The three related pairs of angles can be identified by the letters F, Z and C. The associated properties hold true as long as the two lines are parallel to each other, even if the letters are in a different orientation or laterally inverted.

- The letter F shows corresponding angles.

- The letter Z shows alternate angles.



- The letter C shows interior angles.



In the Introductory Problem, how do painters ensure that the lines drawn to define each parking lot are parallel?

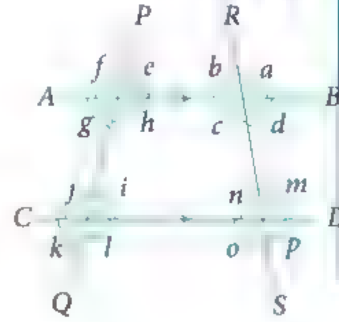
Practise Now 4A

Exercise 10B

In the figure, $AB \parallel CD$.

(a) List
  (i) one pair of equal corresponding angles,
  (ii) one pair of equal alternate angles,
  (iii) one pair of interior angles which are supplementary.

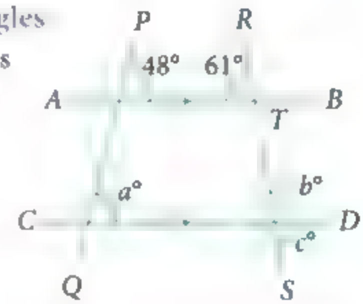
(b) Explain your answer to the following questions.
  (i) Is $\angle e = \angle a$?
  (ii) Is $\angle g = \angle i$?
  (iii) Is $\angle h + \angle c = 180°$?



Worked Example 4

Corresponding angles, alternate angles and interior angles

In the figure, $AB \parallel CD$. T is the point of intersection of the lines AB and RS. Calculate the values of a, b and c.



Solution

$a° = 48°$ (corr. $\angle$s, $AB \parallel CD$)
$\therefore a = 48$

To find value of b:

Method 1:

Then $\angle BTS = 61°$ (vert. opp. $\angle$s)
$b° + \angle BTS = 180°$ (int. $\angle$s, $AB \parallel CD$)
$b° + 61° = 180°$
$b° = 180° - 61°$
$= 119°$
$\therefore b = 119$

Method 2:

Then $\angle ATS = 180° - 61°$ (adj. $\angle$s on a str. line)
$= 119°$
$b° = 119°$ (alt. $\angle$s, $AB \parallel CD$)
$\therefore b = 119$

To find value of c:

Method 1:

$c° = \angle BTS$ (corr. $\angle$s, $AB \parallel CD$)
$= 61°$
$\therefore c = 61$

Method 2:

$c° = 180° - b°$ (adj. $\angle$s on a str. line)
$= 180° - 119°$
$= 61°$
$\therefore c = 61$

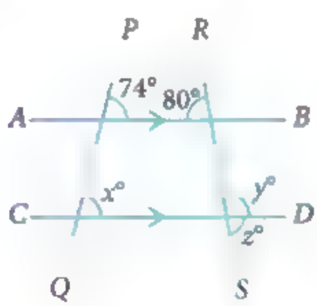
Reflection

Which method do you prefer?
How are these methods related?

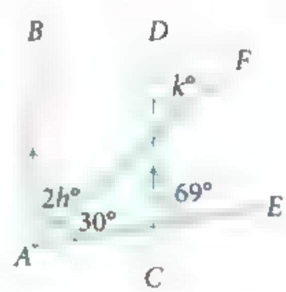
Practise Now 4B

Exercise 10B

1. In the figure, $AB \parallel CD$.
   Find the values of x, y and z.



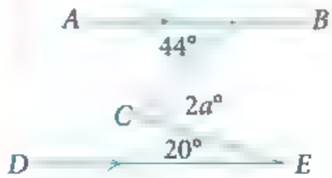
2. In the figure, $AB \parallel CD$.
   Find the values of h and k.

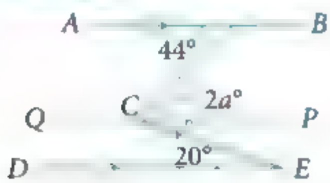


Worked Example 5

Solving more complicated geometrical problem

In the figure, AB // DE. Find the value of a.



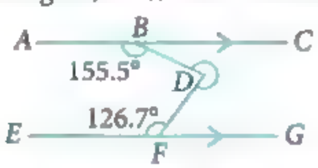


$x° = 44°$ (alt. ∠s, AB // QP)
$y° = 20°$ (alt. ∠s, QP // DE)
$2a° = x° + y°$
$= 44° + 20°$
$= 64°$
$a° = 32°$
$\therefore a = 32$

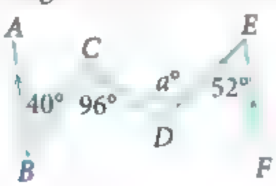
Problem-solving Tip

Draw additional lines if necessary to make use of the properties of the angles formed by parallel lines and transversals to solve problems. In Worked Example 5, we draw a line QP through C that is parallel to AB and DE.

1. In the figure, AC // EG. Find reflex ∠BDF.



2. In the figure, BA // FE. Calculate the value of a.



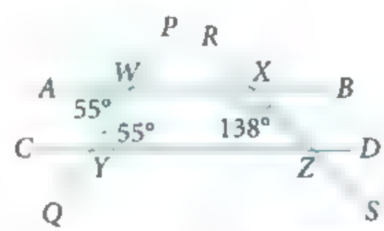
Problem-solving Tip

BD and FD meet at point D where two angles are formed: a non-reflex angle and a reflex angle, which is more than 180°. By convention, ∠BDF refers to the non-reflex angle and we write reflex ∠BDF to refer to the reflex angle.

Worked Example 6

Converse of alternate angles

In the figure, the lines AB and CD are cut by the transversals PQ and RS. If $A\hat{W}Q = D\hat{Y}P = 55°$ and $A\hat{X}S = 138°$, calculate $C\hat{Z}S$.



Solution

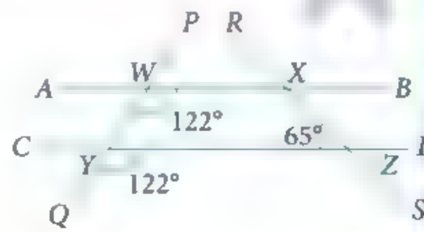
Since $A\hat{W}Q = D\hat{Y}P$ (= 55°),
then AB // CD (converse of alt. ∠s).
$\therefore C\hat{Z}S = A\hat{X}S$ (corr. ∠s, AB // CD)
$= 138°$

Problem-solving Tip

We cannot assume AB // CD. So we need to use the converse property of alt. ∠s to show that AB // CD. Then we can apply the angle property of corr. ∠s to solve for the unknown.

Similar and Further Questions
Exercise 10B
Questions 12, 13

In the figure, the lines AB and CD are cut by the transversals PQ and RS. If $B\hat{W}Q = D\hat{Y}Q = 122°$ and $C\hat{Z}R = 65°$, find $B\hat{X}S$.

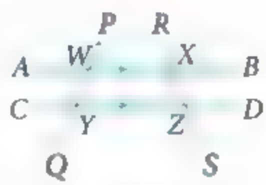


1. What do I already know about parallel lines that could guide my learning in this section?
2. (a) When solving a problem, how do I decide which angle property to use?
   (b) If there is more than one method to solve a problem, how do I determine which method is more efficient?
3. What have I learnt in this chapter that I am still unclear of?

Intermediate Basic

Exercise 10B

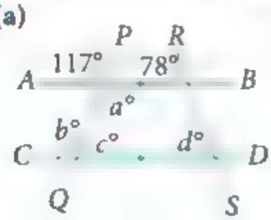
1. In the figure, AB // CD.



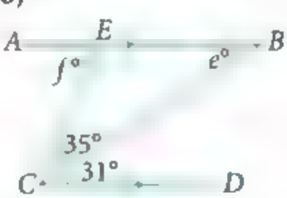
(a) List
   (i) two pairs of equal corresponding angles,
   (ii) two pairs of equal alternate angles,
   (iii) two pairs of interior angles which are supplementary.
(b) Is $B\hat{W}Q = A\hat{X}R$? Explain your answer.
(c) Is the sum of $D\hat{Y}P$ and $C\hat{Z}R$ equal to 180°? Explain your answer.

2. Given that AB // CD, find the values of the unknowns in each of the following figures.

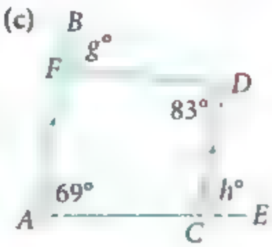
(a)



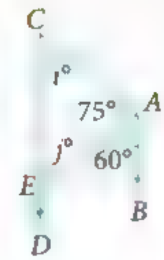
(b)



(c)

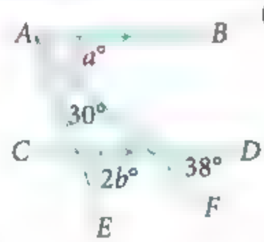


(d)

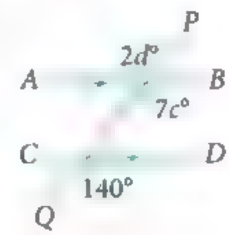


3. In each of the following figures, AB // CD. Find the value(s) of the unknown(s).

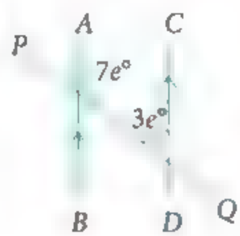
(a)



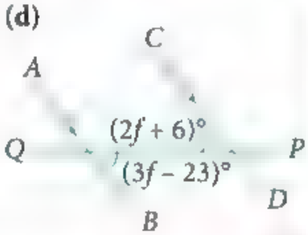
(b)



(c)



(d)



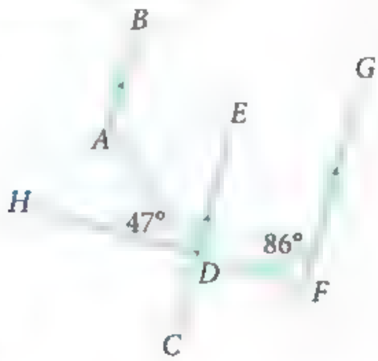
Exercise 10B

4. In the figure, $AB // CD$, $BF // AD$, $E\hat{B}F = 68°$ and $C\hat{D}E = 58°$.

Find
(i) $A\hat{E}B$,
(ii) $A\hat{B}E$.

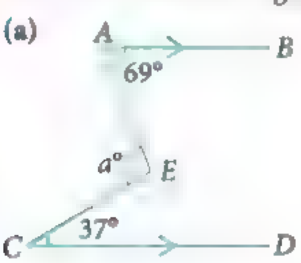
5. In the figure, HDF is a straight line, $AB // CE // FG$, $A\hat{D}H = 47°$ and $D\hat{F}G = 86°$.

Find
(i) $C\hat{D}F$,
(ii) $B\hat{A}D$.

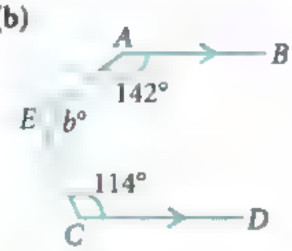


6. Given that $AB // CD$, find the value of the unknown in each of the following figures.

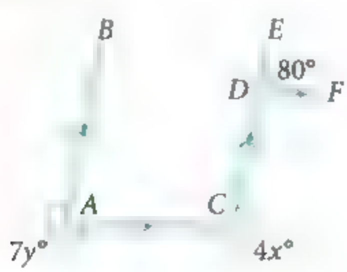
(a)



(b)

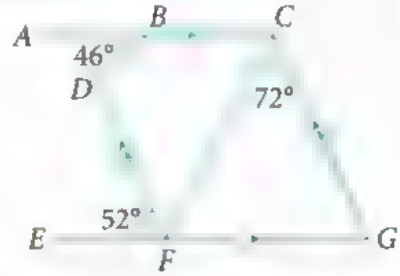


7. In the figure, $AB // CE$ and $DF // AC$. Find the value of x and of y.

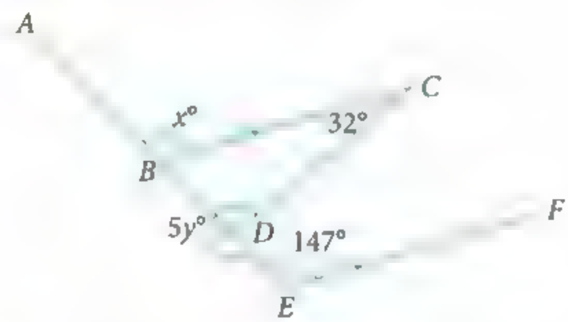


8. In the figure, $AC // EG$, $FD // GC$, $A\hat{B}D = 46°$, $D\hat{F}E = 52°$ and $F\hat{C}G = 72°$.

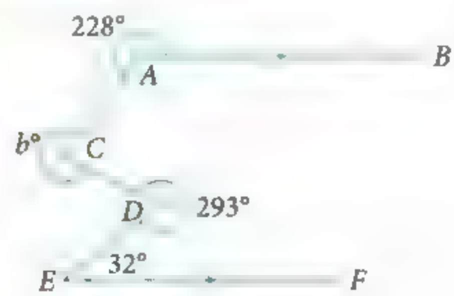
Find
(i) $C\hat{G}F$,
(ii) $B\hat{C}F$,
(iii) reflex $B\hat{D}F$.



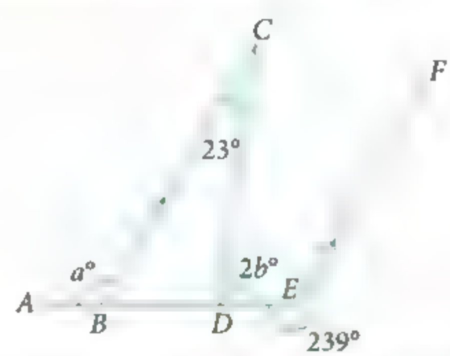
9. In the figure, $ABDE$ is a straight line and $BC // EF$. Find the value of x and of y.



10. In the figure, $AB // EF$. Find the value of b.

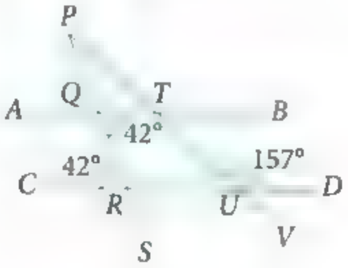


11. In the figure, $ABDE$ is a straight line and $BC // EF$. Calculate the value of a and of b.

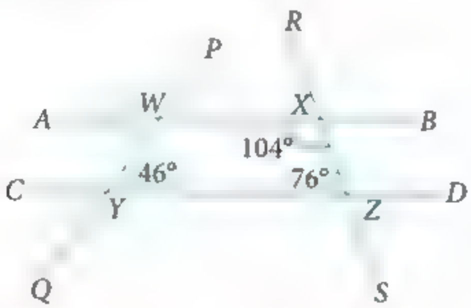


Exercise 10B

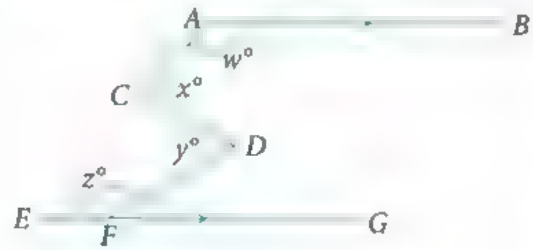
12. In the figure, the lines AB and CD are cut by the transversals PS and PV. If $\angle CRQ = \angle RQT = 42°$ and $\angle TUD = 157°$, find $\angle PTQ$.



13. In the figure, the lines AB and CD are cut by the transversals PQ and RS. If $D\hat{Y}P = 46°$, $A\hat{X}S = 104°$ and $C\hat{Z}R = 76°$, find $B\hat{W}P$.

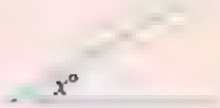
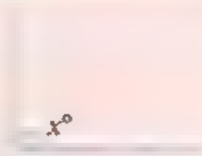
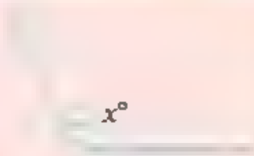
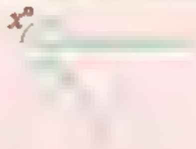


In the figure, $AB \parallel EG$, $B\hat{A}C = w°$, $A\hat{C}D = x°$, $C\hat{D}F = y°$ and $D\hat{F}E = z°$. Form an equation connecting w, x, y and z.



In this chapter, we revisited some key ideas that we have learnt in primary school (e.g. point, line and angle) and learnt some new properties involving angles. It may surprise you, but geometrical objects like a point (with zero dimension) or a line (an object with length but no breadth) exist only in our minds, i.e. we cannot actually see them! This illustrates the power and beauty of mathematics to help visualise the unseen. We can create geometrical objects in our mind, represent them using geometrical diagrams, and measure the attributes of these objects. Diagrams provide a way to describe physical objects in terms of points, lines and shapes. These representations are used to represent real-world situations, enabling us to work with and gain insights to these abstract ideas. Mathematicians also come up with measures of properties so that we can analyse and compare them. For example, an angle is used as a measure of the amount of turn between two lines about a fixed point. The notion of an angle has further applications involving turnings, rotations and revolutions in the real world. Mathematics is beautiful in its applications and in the way it connects our mental and physical worlds!
Pythagoras once said, "There is geometry in the humming of the strings, there is music in the spacing of the spheres." As we marvel at "the humming of the strings", let us remember to appreciate the beauty of mathematics

1. Types of angles

Acute angle	Right angle	Obtuse angle
		
$0° < x° < 90°$	$x° = 90°$	$90° < x° < 180°$
Straight angle	**Reflex angle**	
	 	
$x° = 180°$	$180° < x° < 360°$	

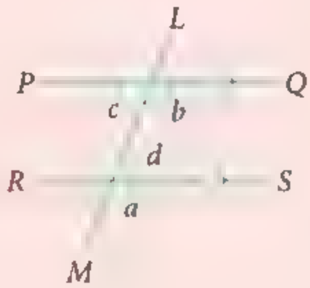
2. **Complementary and supplementary angles**
   - **(a)** Complementary angles are two angles that add up to 90°.
   - **(b)** Supplementary angles are two angles that add up to 180°.
   - Give an example of a pair of complementary angles and a pair of supplementary angles.

3. **Properties of angles formed by intersecting lines**
   - **(a)** The sum of adjacent angles on a straight line is 180° (adj. ∠s on a str. line).
   - **(b)** The sum of angles at a point is 360° (∠s at a point).
   - **(c)** Vertically opposite angles are equal (vert. opp. ∠s).
   - Draw a figure to illustrate each of the above angle properties.

4. **Properties of angles formed by two parallel lines and a transversal**

   If two parallel lines, PQ and RS, are cut by a transversal LM, then
   - *corresponding angles* are equal, e.g. $\angle a = \angle b$ (corr. ∠s, PQ // RS);
   - *alternate angles* are equal, e.g. $\angle c = \angle d$ (alt. ∠s, PQ // RS);
   - *interior angles* are supplementary, e.g. $\angle b + \angle d = 180°$ (int. ∠s, PQ // RS).

   The converse for each of the above is also true.
   - State the converse of each of the above.



CHAPTER 11

Polygons and Geometrical Constructions

Examine the photo of a honeycomb.

Do you notice that every cell is a six-sided figure or a hexagon?

If you look closely enough, you will see that each cell has six equal sides and interior angles. Such hexagons are called regular hexagons. This inspired the Honeycomb Conjecture, proposed by a Roman scholar Marcus Terentius Varro in 36 BC, which states that the hexagonal honeycomb is the best way to divide a surface into regions of equal area with the least total perimeter. Professor Thomas Hale later proved it mathematically in 1999. Thus, the hexagonal cells in a honeycomb enable bees to build a simple compact structure to live in.

Learning Outcomes

What will we learn in this chapter?

- What polygons (including triangles and special quadrilaterals) and their properties (including angle and symmetric properties) are
- How to construct triangles and quadrilaterals using mathematical instruments
- How to solve problems involving the properties of polygons
- Why polygons have useful applications in real life

Introductory Problem

Fig. 11.1

The honeycomb can be modelled by a tessellation of regular six-sided figures called hexagons, as shown in Fig. 11.1.
A hexagon is an example of a polygon (which means many angles in Greek).

1. Show that each interior angle of a regular hexagon is 120°.
2. In this model, we use hexagons as the basic unit for tessellation. What other regular polygons can be used for tessellating the plane without any gaps? Why are these polygons able to form tessellations?
   Hint It has something to do with the interior angles.
3. How can we apply the method used in Question 1 to calculate the interior angle of other regular polygons? What about non-regular polygons?

In this chapter, we will learn how to determine the interior angles of a regular polygon and explore other geometrical properties of polygons.

11.1 Triangles

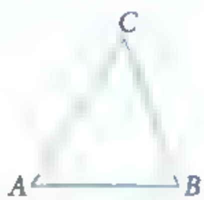


Fig. 11.2

Fig. 11.2 shows a triangle ABC (abbreviation: $\triangle ABC$) that has three sides AB, BC and AC. The points A, B and C are called the vertices (singular: vertex) of the triangle. $\angle BAC$, $\angle ABC$ and $\angle ACB$ are known as the interior angles of $\triangle ABC$. Another way to label the angles is $B\hat{A}C$, $A\hat{B}C$ and $A\hat{C}B$.

A. Classification of triangles

Table 11.1 classifies triangles according to the number of equal sides they have. In primary school, we have learnt what an equilateral triangle and an isosceles triangle are. How many equal sides does an equilateral triangle and an isosceles triangle each have? Fill in the blanks in Table 11.1.

Attention

To indicate equal sides, we draw a tick on each of them, as shown in Table 11.1.

Name	Definition	Figure	Properties
Equilateral triangle	A triangle with ____ equal sides		All the angles in an equilateral triangle are equal, i.e. 60°. (abbreviation: ∠s of equilateral △)
Isosceles triangle	A triangle with at least ____ equal sides		The base angles of an isosceles triangle are equal. (abbreviation: base ∠s of isos. △)
Scalene triangle	A triangle with no equal sides		All the angles in a scalene triangle are different.

Table 11.1

Information

Euclid defined an isosceles triangle to have exactly 2 equal sides (this is an exclusive definition as it excludes the equilateral triangle).
Nowadays, many people use the inclusive definition of an isosceles triangle, i.e. an isosceles triangle has at least 2 equal sides, to include the equilateral triangle. Therefore, based on the inclusive definition, an equilateral triangle is a special type of isosceles triangle.

Table 11.2 classifies triangles according to the types of angles they have. In Chapter 10, we have recapped that an acute angle is between 0° and 90°, a right angle is equal to 90°, and an obtuse angle is between 90° and 180°

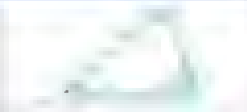
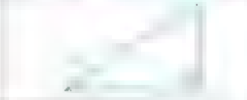
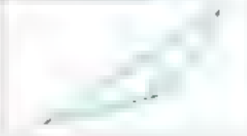
Name	Definition	Figure
Acute-angled triangle	A triangle with 3 acute angles	
Right-angled triangle	A triangle with a right angle	
Obtuse-angled triangle	A triangle with an obtuse angle	

Table 11.2

B. Basic properties of triangle

Basic properties of triangle

Let us explore two basic properties of a triangle. Go to www sl-education.com/tmsoupp1/pg261 or scan the QR code on the right and open the geometry software template 'Basic Properties of Triangle'.

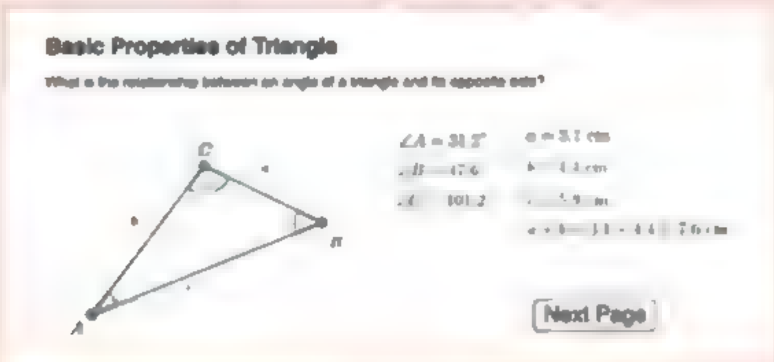


Fig. 11.3

1. The template in Fig 11.3 shows a triangle with 3 angles and their opposite sides, e.g. the side a is opposite to $\angle A$. Name the side opposite to $\angle B$ and the side opposite to $\angle C$.
2. State the largest angle and the smallest angle.
   Compare the lengths of the sides opposite these angles What do you observe?
3. Click and move a point A, B or C to change the size of the triangle. What can you conclude about the relationship between an angle of a triangle and the length of its opposite side?
4. Click and move the point C to adjust the lengths of the sides a and b, while maintaining the side c as the longest side. What do you notice about the sum of the lengths of the two shorter sides of a triangle compared to the length of the longest side?
5. Repeat Step 3. Is your observation in Step 4 still applicable? Explain your answer.

Click the 'Next Page' button at the bottom right of the template to proceed to the next page as shown in Fig. 11.4.

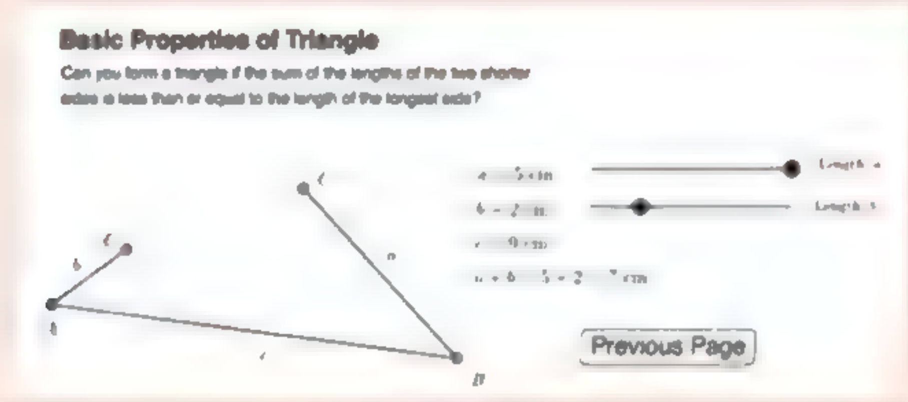


Fig. 11.4

6. The template in Fig. 11.4 shows three line segments with lengths $a = 5$ cm, $b = 2$ cm and $c = 9$ cm.
   Notice that the sum of the lengths of the two shorter line segments, i.e. $a + b$, is shorter than the length of the longest line segment.
   Click and move the two points labelled C to see if it is possible to form a triangle.
7. Adjust the lengths a and b by moving the two sliders at the side of the template so that $a = 3$ cm and $b = 4$ cm. Adjust the length c by clicking and moving either point A or B such that $c = 7$ cm.
   What do you notice about $a + b$ and c? Try to form a triangle if possible.
8. Change the lengths of the three line segments. Are you able to form a triangle?
   What can you conclude about the relationship between the sum of the lengths of any two line segments and the length of the third line segment?

From the above Investigation, we observe the following:

Basic properties of triangle

- The largest angle of a triangle is opposite the longest side, and the smallest angle is opposite the shortest side.
- The sum of the lengths of any two (shorter) sides of a triangle must be greater than the length of the third side.

Invariance

In the above Investigation, no matter how you change the shape or size of the triangle, these two basic properties remain the same. We say that the properties are invariant. Invariance refers to a property of a mathematical object which remains unchanged when the object undergoes some form of transformation.

C. Angle sum of triangle

Look at the triangle in Fig. 11.3 in the Investigation on page 261. What is the sum of $\angle A$, $\angle B$ and $\angle C$ in the triangle? Is this true for all triangles?

Copy to complete the following. Consider $\triangle ABC$ in Fig. 11.5.

Draw a line PQ passing through C and parallel to AB.

$\angle BAC = \angle ACP$ (alt. $\angle$s, $PQ \,//\, AB$)

$\angle ABC = \angle$ ___ (___ $\angle$s, $PQ \,//$ ___)

Since $\angle ACP + \angle ACB + \angle BCQ =$ ___ (adj. $\angle$s on a str. line),

then $\angle BAC + \angle ACB + \angle$ ___ $=$ ___ ($\angle$ sum of $\triangle$).

Fig. 11.5

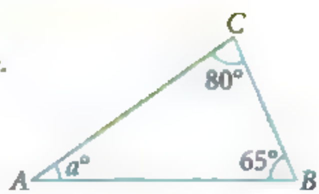
From the above Thinking Time, we have shown the following:

Angle sum of triangle

The sum of interior angles of a triangle is 180°.
(abbreviation: $\angle$ sum of $\triangle$)

Angle sum of triangle

Find the value of a in the figure.



Solution

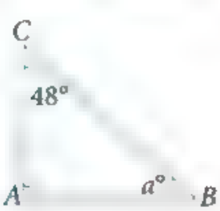
$a° + 65° + 80° = 180°$ ($\angle$ sum of $\triangle$)

$a° = 180° - 65° - 80°$

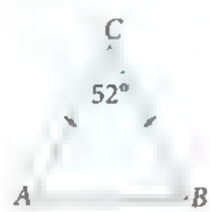
$= 35°$

$\therefore a = 35$

1. Find the value of a in the figure.



2. In the figure, $AC = BC$. Calculate $\angle ABC$.



Problem-solving Tip

2. The base angles of an isosceles triangle are equal (abbreviation: base $\angle$s of isos. $\triangle$)

D. Exterior angles of triangle

Fig. 11.6(**a**) shows $\triangle ABC$ with AB produced to P, BC produced to Q and CA produced to R. $\angle a$, $\angle b$ and $\angle c$ are the interior angles of $\triangle ABC$, while $\angle p$, $\angle q$ and $\angle r$ are the **exterior angles** of $\triangle ABC$.
Notice that the exterior angles go in an anticlockwise direction from $\angle p$ to $\angle q$ to $\angle r$.
Fig. 11.6(**b**) shows the exterior angles of $\triangle ABC$ drawn in a clockwise direction. Although there are two ways to draw the exterior angles of a triangle, note that a triangle has *exactly 3 exterior angles*, just like it has exactly 3 interior angles.

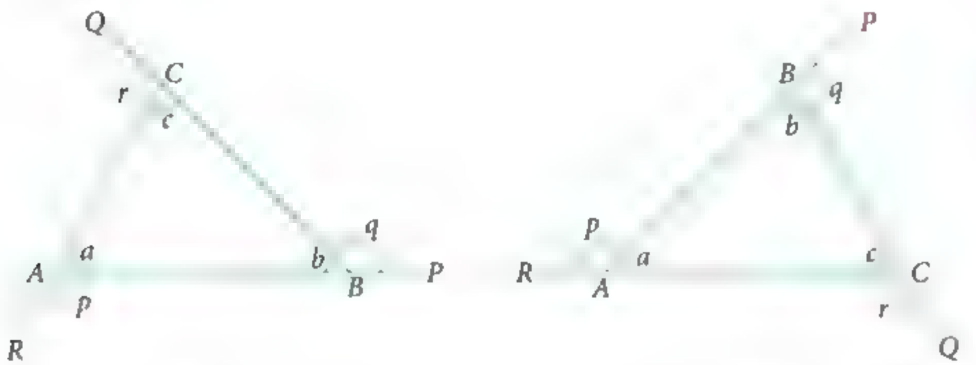


Fig. 11.6

Attention

This is *not* an exterior angle of a triangle:

$\angle a$ and $\angle b$ are called the **interior opposite angles** with reference to the exterior $\angle r$.
Similarly, $\angle a$ and $\angle c$ are called the interior opposite angles with reference to the exterior $\angle q$.
Which are the interior opposite angles with reference to the exterior $\angle p$?

How is an exterior angle of a triangle related to its interior opposite angles? Let us investigate.
Copy to complete the following.
Consider $\triangle ABC$ in Fig. 11.7.

$\angle b + \angle d =$ ______ (adj. $\angle$s on a str. line)

$\angle a + \angle b + \angle c =$ ______ ($\angle$ sum of $\triangle$)

$\therefore \angle b + \angle d = \angle a + \angle b + \angle$ ______

$\therefore$ exterior $\angle d = \angle$ ______ $+ \angle$ ______

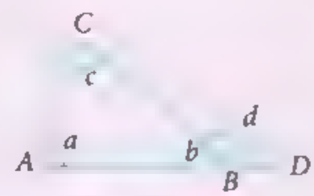


Fig. 11.7

From the above Thinking Time, we have proven the following:

Exterior angle of triangle

An exterior angle of a triangle is equal to the sum of its interior opposite angles. (abbreviation: ext. $\angle$ of $\triangle$)

From Section 11.1C and the above Thinking Time, we note that no matter how the shape or size of a triangle is changed, the angle properties of a triangle are invariant.

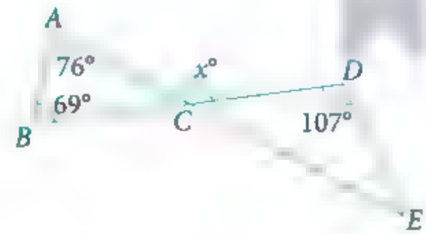
2

Exterior angle of triangle

In the diagram, ACE and BCD are straight lines.

(a) Find the value of x.

(b) Calculate $\angle CED$.



Solution

(a) $x° = 76° + 69°$ (ext. $\angle$ of $\triangle$)

$= 145°$

$\therefore x = 145$

(b) $\angle CED + \angle CDE = x°$ (ext. $\angle$ of $\triangle$)

$\angle CED + 107° = 145°$

$\angle CED = 145° - 107°$

$= 38°$

Reflection

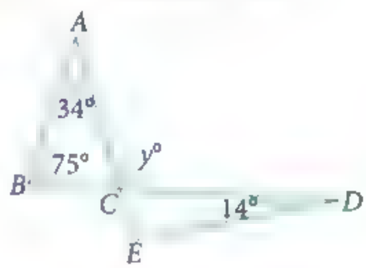
(a) What is another way to solve for x? Which way do you prefer?

Similar and Further Questions

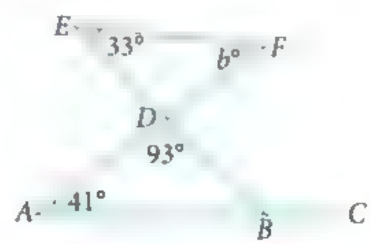
Exercise 11A

Question:

1. In the diagram, ACE and BCD are straight lines.
   (a) Find the value of y.
   (b) Calculate $\angle CED$.



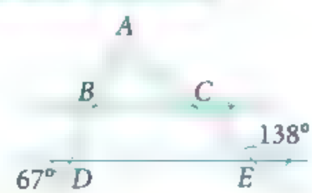
2. In the figure, ABC, ADF and BDE are straight lines.
   (a) Calculate $\angle CBD$.
   (b) Find the value of b.



3

[illegible]

In the diagram, BC // DE. Calculate $D\hat{A}E$.



Solution

$B\hat{D}E = 67°$ (vert. opp. $\angle$s)

$A\hat{B}C = B\hat{D}E$ (corr. $\angle$s, BC // DE)

$= 67°$

$B\hat{C}E = 138°$ (alt. $\angle$s, BC // DE)

$\therefore D\hat{A}E = B\hat{C}E - A\hat{B}C$ (ext. $\angle$ of $\triangle$)

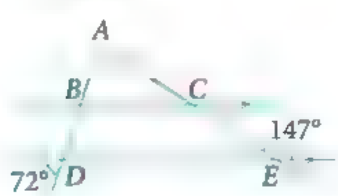
$= 138° - 67°$

$= 71°$

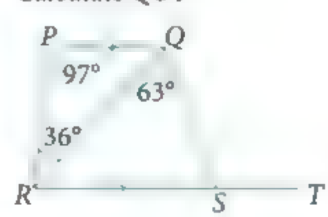
Reflection

What is another way to solve for $D\hat{A}E$? Which way do you prefer?

1. In the diagram, BC // DE. Calculate $D\hat{A}E$.



2. In the figure, PQ // RT. Calculate $Q\hat{S}T$.



1. What do I already know about parallel lines in Chapter 10 that could guide my learning of the properties of triangles in this section?
2. What have I learnt in this section that I am still unclear of?

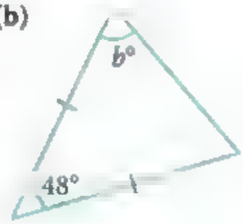
Basic | Intermediate

Exercise 11A

1. Find the value of the unknown in each of the following figures, correct to 1 decimal place where appropriate.

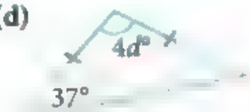
(a)

(b)



(c)

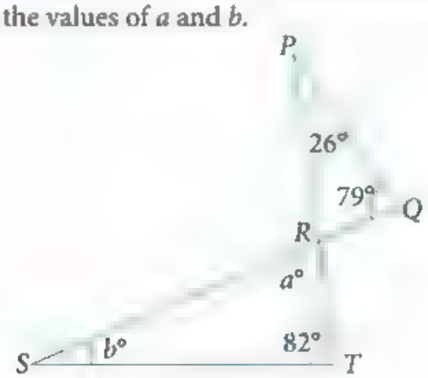
(d)



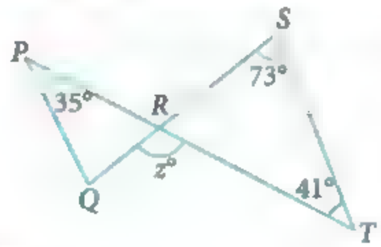
(e)

(f)

2. In the diagram, PRT and QRS are straight lines. Find the values of a and b.

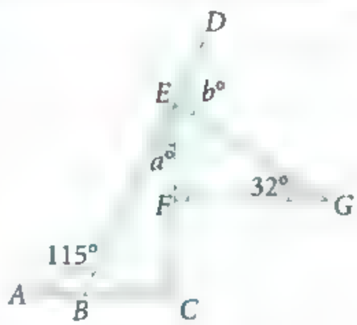


3. In the figure, PRT and QRS are straight lines.

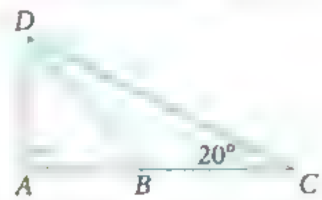


(i) Find the value of z.
(ii) Calculate $\angle PQS$.

4. Given that ABC and BED are straight lines, find the values of the unknowns in the diagram.



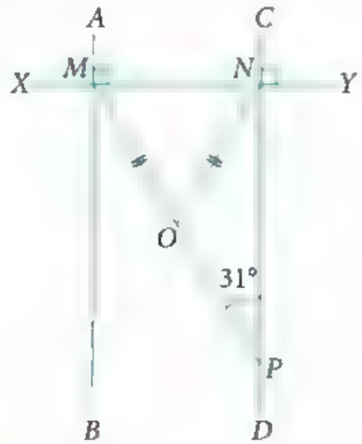
5. In the figure, ABC is a straight line.



Given that $A\hat{D}B = B\hat{D}C$, calculate
(i) $B\hat{D}C$,
(ii) $C\hat{B}D$.

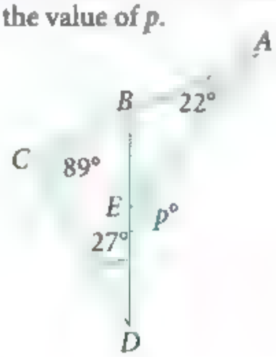
Exercise 11A

6. In the figure, the lines AB and CD are cut by the transversal XY. MOP is a straight line. If $OM = ON$, $A\hat{M}N = C\hat{N}Y = 90°$ and $O\hat{P}N = 31°$, find $M\hat{O}N$.

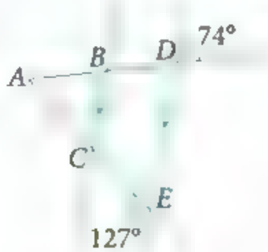


7. In $\triangle ABC$, $\angle BAC = 53°$ and $\angle ACB = 28°$. If AB is produced to D, find $\angle CBD$.

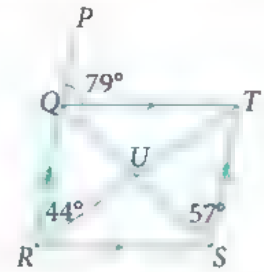
8. Given that ABC and BED are straight lines in the figure, find the value of p.



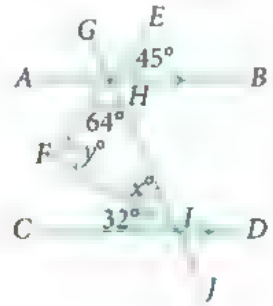
9. In the figure, BC // DE. Calculate $D\hat{A}E$.



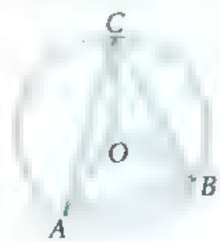
10. In the figure, QT // RS, RQ // ST and PQR, QUS and RUT are straight lines. Calculate $S\hat{U}T$ and $R\hat{S}U$.



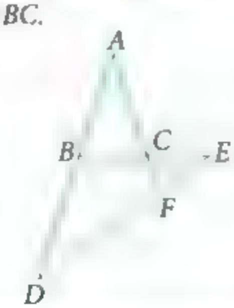
11. In the figure, AB // CD, and $GHIJ$ is a straight line. Find the values of the unknowns.



12. The figure shows $\triangle ABC$ inscribed in a circle with centre O. If $\angle CBO$ is twice of $\angle CAO$ and $\angle BAO$ is one and a half times of $\angle CBO$, calculate $\angle CAO$.



13. In the diagram, each side of $\triangle ABC$ is produced. If $AB = AC$, $BD = BE$ and $AF = DF$, calculate $\angle ABC$.



11.2 Quadrilaterals

In primary school, we have learnt about rectangles, squares, parallelograms, rhombuses and trapeziums. These 4-sided figures are called **quadrilaterals**.

Fig. 11.8 shows a quadrilateral *ABCD*, which is a closed plane figure with 4 vertices, 4 straight line segments as its sides and 4 interior angles. Two of the interior angles are $\angle DAB$ and $\angle ABC$. Name the other two interior angles.

The line segment *BD* that joins the *non-adjacent* vertices, *B* and *D*, is called a **diagonal** of the quadrilateral *ABCD*.

Name the other diagonal of the quadrilateral.

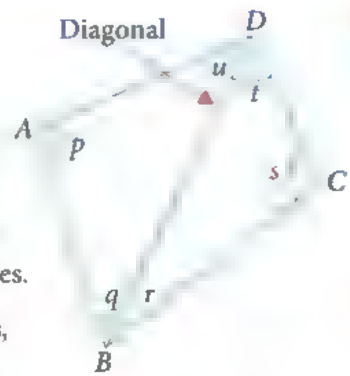


Fig. 11.8

Attention

A quadrilateral is named by taking the vertices either in a clockwise or an anticlockwise order. Hence, *ABCD*, *BCDA*, *CDAB* and *DABC* are correct ways of naming the quadrilateral, but *ABDC* and *BCAD* are not.

A. Angle sum of quadrilateral

In Section 11.1, we have learnt that the sum of interior angles of a triangle is 180°. What about the sum of interior angles of a quadrilateral?

The quadrilateral *ABCD* in Fig. 11.8 can be divided into two triangles: $\triangle ABD$ and $\triangle BCD$.

Copy to complete the following proof.

$\angle p + \angle q + \angle u = 180°$ ($\angle$ sum of $\triangle ABD$)

$\angle r + \angle s + \angle t = \underline{\qquad}$ ($\angle$ sum of $\triangle \underline{\qquad}$)

Sum of interior angles of the quadrilateral $ABCD = \angle p + \angle q + \angle r + \angle s + \angle t + \angle \underline{\qquad}$

$= (\angle p + \angle q + \angle u) + (\angle r + \angle \underline{\qquad} + \angle \underline{\qquad})$

$= 180° + \underline{\qquad}$

$= \underline{\qquad}$

From the above Thinking Time, we conclude that:

Angle sum of quadrilateral

The sum of the interior angles of a quadrilateral is 360°.
(abbreviation: $\angle$ sum of quad.)

B. Properties of special quadrilaterals

Rectangles, squares, parallelograms, rhombuses and trapeziums are examples of special quadrilaterals with defining properties. In this section, we will learn more about their properties, and another special quadrilateral called a kite.

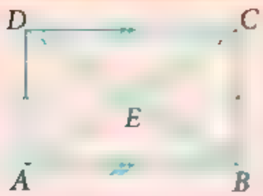
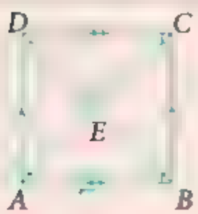
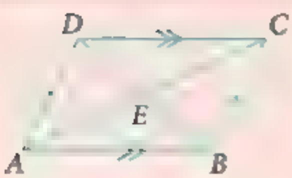
Properties of special quadrilaterals

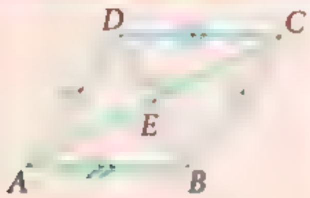
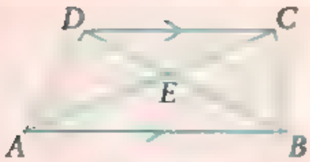
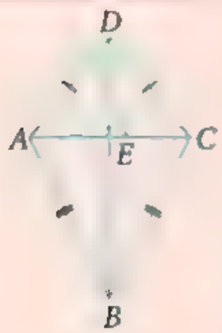
Let us explore the properties of special quadrilaterals.

Go to www.sl-education.com/tmsoupp1/pg269 or scan the QR code on the right and open the geometry software template 'Special Quadrilaterals'.

To complete Table 11.3, use the template to make some measurements for each quadrilateral, e.g. measure the length of each side and diagonal.

Hint: The word '**bisect**' means 'divide into two equal parts'.

Quadrilateral	Parallel sides	Equal sides	Interior angles	Diagonals
 **Rectangle**	There are pairs of parallel sides.	Opposite sides are equal in length.	All four angles are right angles.	• The two diagonals are in length. • Diagonals *bisect* each other, i.e. $AE = EC$ and $BE = ED$.
 **Square**	There are two pairs of parallel sides.	All sides are equal in length.	All angles are right angles.	• The two diagonals are equal in length. • Diagonals *bisect* each other at right angles, i.e. $AE = EC$, $BE = ED$ and $\angle AEB = \angle BEC = \angle CED = \angle AED = 90°$. • Diagonals the interior angles, e.g. $\angle BAC = \angle CAD = 45°$ and $\angle ABD = \angle CBD = 45°$.
 **Parallelogram**	There are pairs of parallel sides.	sides are equal in length.	angles are equal, i.e. $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$.	Diagonals each other, i.e. $AE = EC$ and $BE = ED$.

Quadrilateral	Parallel sides	Equal sides	Interior angles	Diagonals
 **Rhombus**	There are ____ pairs of parallel sides.	All four sides are equal in length.	____ angles are equal, i.e. $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$.	• Diagonals ____ each other at right angles, i.e. $AE = EC$, $BE = ED$ and $\angle AEB = \angle BEC = \angle AED = \angle CED = 90°$. • Diagonals ____ the interior angles, e.g. $\angle BAC = \angle CAD$ and $\angle ABD = \angle CBD$.
 **Trapezium**	There is at least* one pair of parallel sides, i.e. $AB // DC$.			
 **Kite**		There are at least** two pairs of equal adjacent sides, i.e. $AD = DC$ and $AB = BC$.		• Diagonals cut each other (*not bisect*) at ____ angles, i.e. $\angle AEB = \angle BEC = \angle AED = \angle CED = 90°$. • One diagonal ____ the interior angles, i.e. $\angle ADB = \angle CDB$ and $\angle ABD = \angle CBD$.

* We will use the *inclusive definition* of a trapezium in this textbook. A trapezium has *at least one pair* of parallel sides. If a trapezium has two pairs of parallel sides, then it is a parallelogram (special type of trapezium).
(Recall: We also use the inclusive definition of an isosceles triangle in Section 11.1. An isosceles triangle has at least two equal sides.)

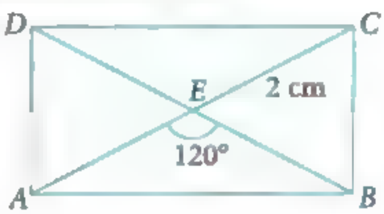
** Similarly, we will also use the inclusive definition of a kite in this textbook. A kite has *at least two pairs* of equal adjacent sides. If a kite has four pairs of equal adjacent sides, this means that all its sides are equal in length, i.e. it is a rhombus (special type of kite). Can a kite have three pairs of equal adjacent sides?

Table 11.3

Worked Example 4

Angles in rectangle

The diagram shows a rectangle $ABCD$ where diagonals AC and BD intersect at E.
Given that $\angle AEB = 120°$ and $CE = 2$ cm,



calculate
(i) $\angle ABE$,
(ii) $\angle BDC$,
(iii) the length of BC.

Solution

(i) Since the diagonals of the rectangle bisect each other, then $AE = BE$.

$\therefore \angle ABE = \dfrac{180° - 120°}{2}$ (base $\angle$s of isos. $\triangle ABE$)

$= 30°$

(ii) $\angle BDC = \angle ABE$ (alt. $\angle$s, $AB \parallel DC$)

$= 30°$

(iii) $\angle CBE = 90° - \angle ABE$ (int. $\angle$ of rect.)

$= 90° - 30°$

$= 60°$

$\angle BCE = \angle CBE$ (base $\angle$s of isos. $\triangle BCE$)

$= 60°$

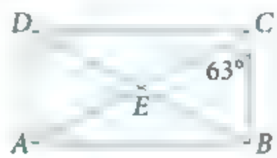
$\angle BEC = 180° - 120°$ (adj. $\angle$s on a str. line)

$= 60°$

Since $\angle CBE = \angle BCE = \angle BEC = 60°$, then $\triangle BCE$ is an equilateral triangle.

$\therefore BC = CE = 2$ cm

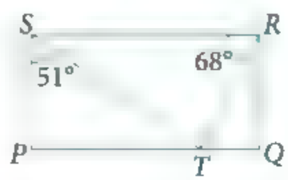
1. The diagram shows a rectangle $ABCD$ where the diagonals AC and BD intersect at E. Given that $\angle ACB = 63°$,



find

(i) $\angle BEC$, **(ii)** $\angle CDE$.

2. The diagram shows a rectangle $PQRS$. T lies on PQ such that $\angle PST = 51°$ and $\angle SRT = 68°$.

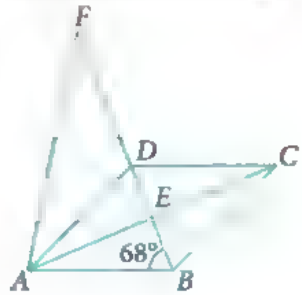


Find

(i) $\angle PTS$, **(ii)** $\angle RTS$.

Angles in rhombus

The diagram shows a rhombus $ABCD$ where diagonals AC and BD intersect at E. The diagonal BD is produced to F such that $AD = DF$. If $A\hat{B}E = 68°$,



calculate

(i) $B\hat{C}D$,

(ii) $D\hat{A}F$,

(iii) $E\hat{A}F$.

(i) $C\hat{B}D = A\hat{B}D$ (diagonal BD of rhombus bisects $A\hat{B}C$)

$= 68°$

$B\hat{C}D + A\hat{B}D + C\hat{B}D = 180°$ (int. $\angle$s, AB // DC)

$B\hat{C}D + 68° + 68° = 180°$

$B\hat{C}D = 180° - 68° - 68°$

$= 44°$

(ii) $A\hat{D}B = 68°$ (base $\angle$s of isos. $\triangle ABD$; or alt. $\angle$s, AD // BC)

$D\hat{A}F + A\hat{F}D = 68°$ (ext. $\angle$ of $\triangle ADF$)

$D\hat{A}F = A\hat{F}D = \dfrac{68°}{2}$ (base $\angle$s of isos. $\triangle ADF$)

$= 34°$

(iii) $A\hat{E}F = 90°$ (diagonals of rhombus bisect each other at right angles)

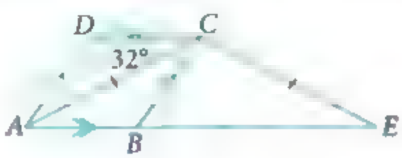
$E\hat{A}F + A\hat{E}F + A\hat{F}E = 180°$ ($\angle$ sum of $\triangle AEF$)

$E\hat{A}F + 90° + 34° = 180°$

$E\hat{A}F = 180° - 90° - 34°$

$= 56°$

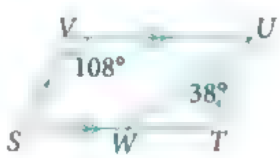
1. The diagram shows a rhombus $ABCD$ where $A\hat{C}D = 32°$. AB is produced to E such that $AC = CE$.



Find

(i) $A\hat{B}C$, (ii) $B\hat{C}E$.

2. The diagram shows a parallelogram $STUV$ where $S\hat{V}U = 108°$. W lies on ST such that $T\hat{U}W = 38°$.

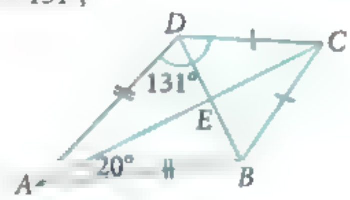


(i) Given that $S\hat{T}U = 9x°$, find the value of x.

(ii) Find $V\hat{U}W$.

Angles in kite

The diagram shows a kite $ABCD$ where $AB = AD$, $CB = CD$ and the diagonals AC and BD intersect at E. If $\angle BAE = 20°$ and $\angle ADC = 131°$,



calculate
(i) $\angle DAE$,
(ii) $\angle CBD$.

Solution

(i) $\angle DAE = \angle BAE$ (longer diagonal AC of kite bisects $\angle BAD$)
$= 20°$

(ii) $\angle BEA = 90°$ (diagonals of kite cut each other at right angles)
$\angle ABE = 180° - \angle BEA - \angle BAE$ ($\angle$ sum of $\triangle ABE$)
$= 180° - 90° - 20°$
$= 70°$
$\angle ABC = \angle ADC$ (opp. $\angle$s of kite)
$= 131°$
$\therefore \angle CBD = \angle ABC - \angle ABE$
$= 131° - 70°$
$= 61°$

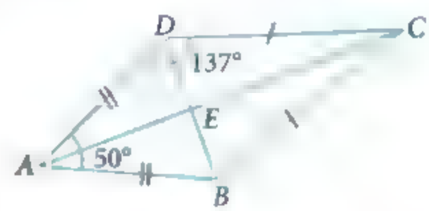
Reflection

(ii) There are at least two other ways of calculating $\angle CBD$. Can you find the alternative ways?

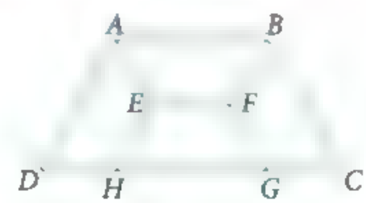
Practise Now 6

Exercise 11B

1. The diagram shows a kite $ABCD$ where $AB = AD$, $BC = DC$ and the diagonals AC and BD intersect at E. If $\angle BAD = 50°$ and $\angle ADC = 137°$. Calculate
(i) $\angle DAE$, (ii) $\angle CDE$.



2. In the diagram, the big trapezium $ABCD$ is divided into four smaller identical trapeziums.
(i) Show that $DH = GC$.
Hint: Consider EH and FG in identical trapeziums $AEHD$ and $BFGC$ respectively. Then consider EH and FG in trapezium $EFGH$.
(ii) Given that $AB = 4$ cm and $DC = 8$ cm, find the length of EF.

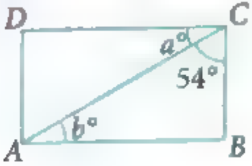


1. What do I already know about the properties of triangles that could guide my learning of the properties of quadrilaterals in this section?
2. What have I learnt in this section that I am still unclear of?

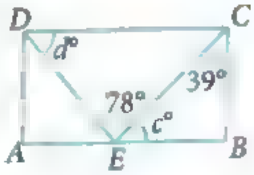
Exercise 11B

1. Find the values of the unknowns in each of the following rectangles.

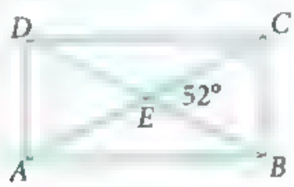
(a)



(b)

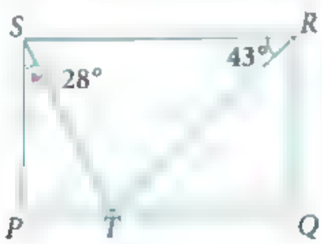


2. The diagram shows a rectangle $ABCD$ where the diagonals AC and BD intersect at E. Given that $\angle BEC = 52°$, calculate



(i) $\angle ADB$, (ii) $\angle ACD$.

3. The diagram shows a rectangle $PQRS$. T lies on PQ such that $\angle PST = 28°$ and $\angle SRT = 43°$.



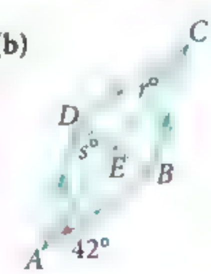
Find

(i) $\angle PTS$, (ii) $\angle RTS$.

4. Find the values of the unknowns in each of the following rhombuses.

(a)

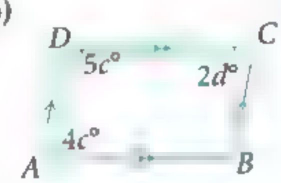
(b)



5. Find the values of the unknowns in each of the following parallelograms.

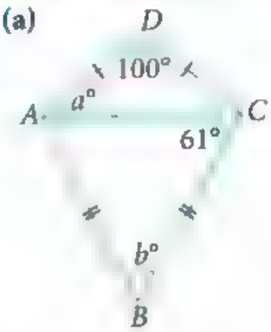
(a)

(b)

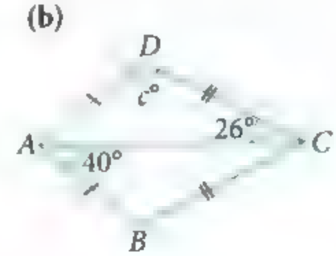


6. Find the value(s) of the unknown(s) in each of the following kites.

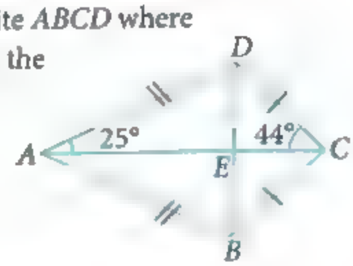
(a)



(b)



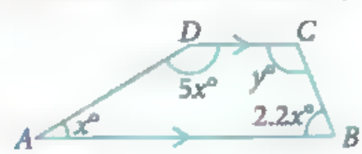
7. The diagram shows a kite $ABCD$ where $AB = AD$, $CB = CD$ and the diagonals AC and BD intersect at E.



Find

(i) $\angle ABD$, (ii) $\angle CBD$.

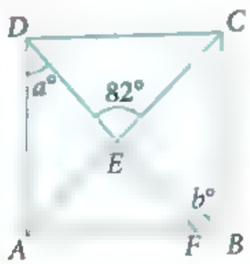
8. The diagram shows a trapezium $ABCD$ where $AB // DC$. Find the value of x and of y.



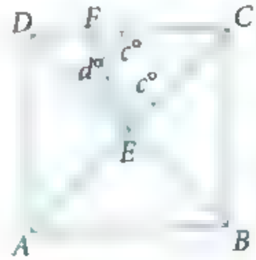
Exercise 11B

9. Find the values of the unknowns in each of the following squares.

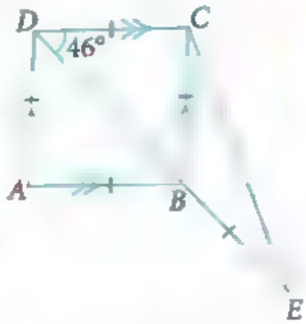
(a)



(b)



10. The diagram shows a rhombus $ABCD$. The diagonal DB is produced to E such that $BC = BE$ and $C\hat{D}E = 46°$. Calculate
(i) $B\hat{A}D$, (ii) $B\hat{C}E$.



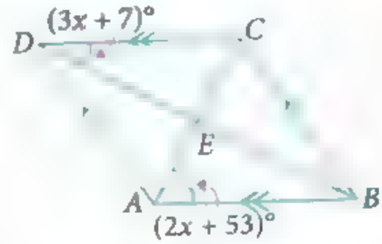
11. The diagram shows a parallelogram $ABCD$ where $B\hat{A}D = 65°$. E lies on DC such that $B\hat{E}D = 125°$.

Calculate
(i) $A\hat{D}E$, (ii) $C\hat{B}E$.

12. In a rhombus $WXYZ$, $W\hat{X}Y = 108°$. Calculate
(i) $X\hat{Z}Y$, (ii) $X\hat{Y}Z$,
(iii) $X\hat{W}Y$.

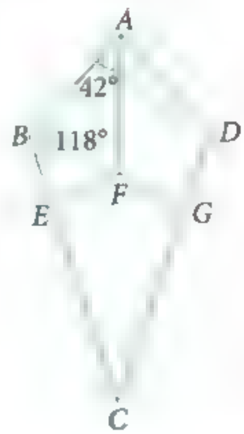
13. In a parallelogram $PQRS$, $Q\hat{P}R = 42°$ and $Q\hat{R}S = 70°$. Calculate
(i) $P\hat{Q}R$, (ii) $P\hat{R}Q$.

14. The diagram shows a rhombus $ABCD$ where the diagonals AC and BD intersect at E. Find the value of x.



15. In a kite $PQRS$, $PQ = QR$, $PS = RS$, $\angle QPR = 42°$ and $\angle PSR = 64°$. Calculate
(i) $\angle PRS$, (ii) $\angle PQR$.

16. In the diagram, the larger kite $ABCD$ is divided into 3 smaller kites. The smaller kites $ABEF$ and $AFGD$ are identical. Given that $B\hat{A}F = 42°$ and $A\hat{B}C = 118°$, calculate $B\hat{C}D$.

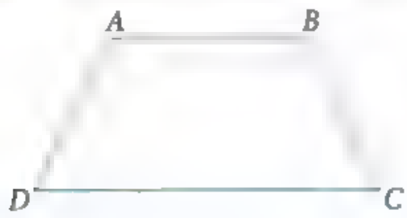


17. In a trapezium $ABCD$, $AB // DC$, $AB = AD$, $\angle BCD = 52°$ and $\angle ADC = 62°$. Calculate
(i) $\angle ABD$, (ii) $\angle CBD$.

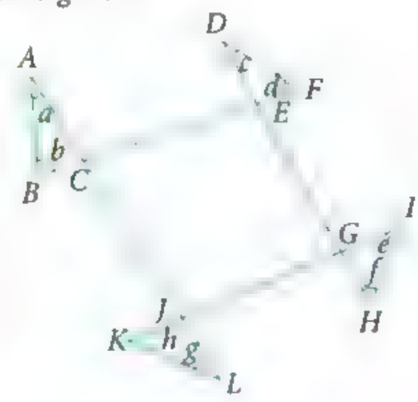
18. In a rectangle $ABCD$, E is the midpoint of AB and $\angle CED = 118°$. Find
(i) $\angle ADE$, (ii) $\angle DCE$.

Exercise 11B

An **isosceles trapezium** is a trapezium where the two non parallel sides are equal.
The diagram shows a trapezium *ABCD* where the base angles $A\hat{D}C$ and $B\hat{C}D$ are equal.
Show that the trapezium *ABCD* is an isosceles trapezium.
Hint: Extend *DA* and *CB* to intersect at *E*.



In the diagram, *ACJL*, *BCEF*, *DEGH* and *IGJK* are straight lines. Calculate $\angle a + \angle b + \angle c + \angle d + \angle e + \angle f + \angle g + \angle h$.



11.3 Geometrical construction of triangles and quadrilaterals

A. Introduction to geometrical constructions

In Chapter 10, we have learnt how to measure angles using a protractor. We will now learn how to construct geometrical figures using a protractor and a pair of compasses. A pair of **compasses** (see Fig. 11.9) is a mathematical instrument consisting of two moveable arms attached together by a hinge. It is used to draw a circle or an arc of a circle, and to mark off a specific length.

Information

A pair of compasses is different from a compass which is used to tell directions. However, in some countries, they use the term 'compass' to refer to a pair of compasses as well.

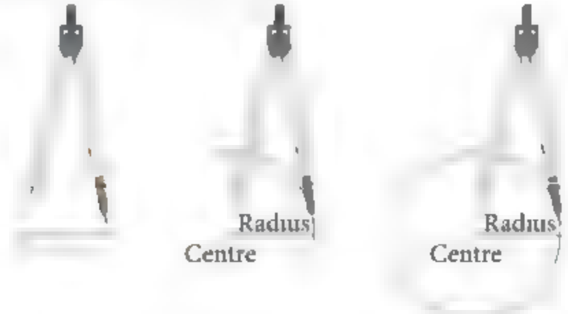
Fig. 11.9

The following show how a pair of compasses can be used.

- Drawing a circle or an arc of a circle

  Step 1 Adjust the arms of the compasses on the markings of a ruler so that the distance between its ends is equal to the specified length, e.g. 7 cm.

  Step 2 Fix the pointed end at the point and move the other arm to draw the circle or an arc of a circle.



- Marking off a length

  Step 1: Adjust the arms of the compasses until the ends touch points A and B (Fig. 11.10(**a**)).

  Step 2: Mark a point P on another line L (Fig. 11.10(**b**)).

  Step 3: Without adjusting the arms of the compasses, fix the pointed end at P and move the other arm to draw an arc cutting L at Q (Fig. 11.10(**c**)).
  Hence, $PQ = AB$.

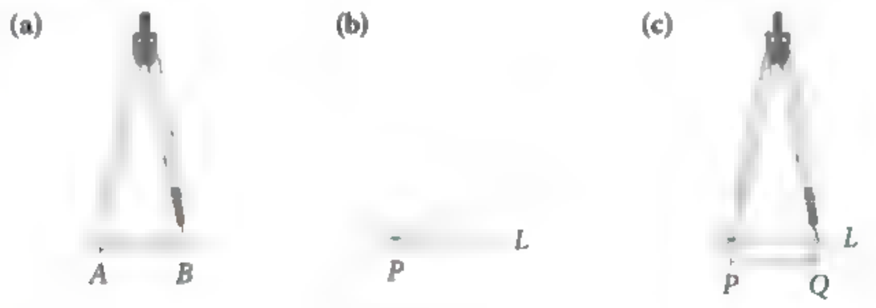


Fig. 11.10

- Constructing perpendicular lines

  Step 1: On a line L, mark the points X and Y. Adjust the arms of the compasses such that the distance between the ends is greater than half the length of XY (Fig. 11.11(**a**)).

  Step 2: Fixed the pointed end at X and move the other arm to draw an arc each above and beneath L (Fig. 11.11(**b**)).

  Step 3: Without adjusting the arms of the compasses, fix the pointed end at Y and move the other arm to draw two arcs to produce the points P and Q (Fig. 11.11(**c**)). Draw a line to connect points P and Q (Fig. 11.11(**d**)). PQ is perpendicular to line L.

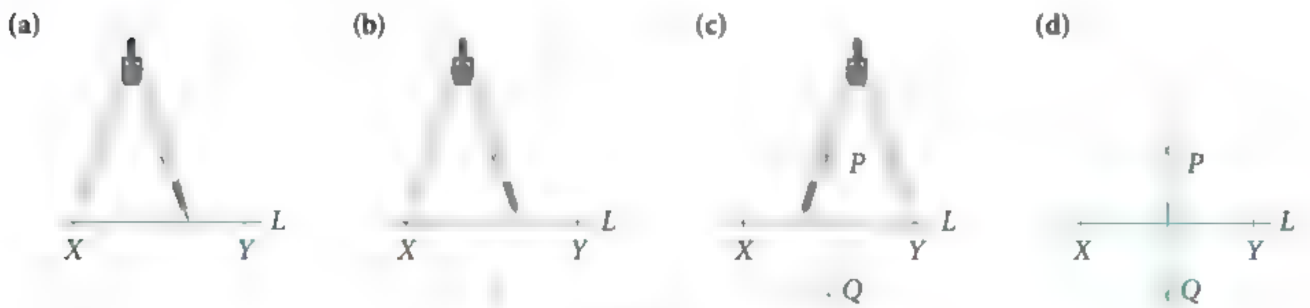


Fig. 11.11

The line PQ constructed using the method described here also bisects XY. That is, if M is the point of intersection between PQ and XY as shown in Fig. 11.12, then XM = MY. We call PQ the perpendicular bisector of XY.

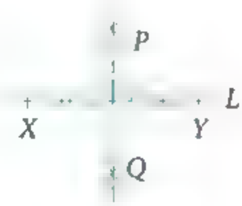


Fig. 11.12

Useful tips for geometrical constructions

The following tips are useful for the construction of geometrical figures:

- Use a sharp pencil so that points and lines can be drawn finely and clearly.
- When making an intersection with a line or an arc, ensure that the angle of intersection does not differ greatly from 90° if possible (see Fig. 11.13).

(a) Good **(b)** Bad

Fig. 11.13

- Be careful when drawing a line through a point to ensure accuracy (see Fig. 11.14).

(a) Good **(b)** Bad

Fig. 11.14

- All construction lines must be clearly shown. Do not erase the construction lines that have been drawn.

B. Construction of triangles

In this section, we will learn how to construct triangles using a protractor and a pair of compasses and solve related problems.

Example 7

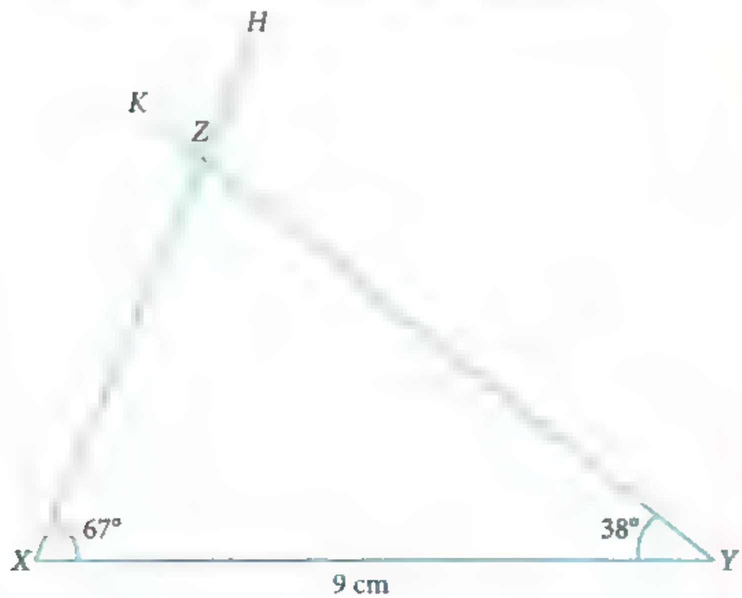
Constructing triangles given 1 side and 2 angles

Construct $\triangle XYZ$ such that $XY = 9$ cm, $X\hat{Y}Z = 38°$ and $Y\hat{X}Z = 67°$.
Measure and write down the length of the shortest side.

Solution

Construction steps:

1. Using a ruler, draw $XY = 9$ cm.
2. Since $\angle X = 67°$, using a protractor at X, mark off an angle of $67°$ and draw a line XH such that $Y\hat{X}H = 67°$.
3. Since $\angle Y = 38°$, using a protractor at Y, mark off an angle of $38°$ and draw a line YK such that $X\hat{Y}K = 38°$.



$\therefore$ length of shortest side $XZ = 5.7$ cm

smallest angle XYZ

Reflection

If both angles are drawn below the line XY, will we obtain the same triangle? If we had drawn the line XY such that Y is the left endpoint instead of X, will we obtain the same triangle?

Attention

Label the vertices, given angles and lengths clearly on the diagram.

Problem-solving Tip

We can read up to the smallest marking on a ruler, which is 0.1 cm. If the measurement is somewhere between 5.7 cm and 5.8, decide whether it is nearer to 5.7 cm or 5.8 cm. If it is nearer to 5.7 cm, write your answer as 5.7 cm.

Construct $\triangle XYZ$ such that $XY = 8$ cm, $X\hat{Y}Z = 56°$ and $Y\hat{X}Z = 48°$. Measure and write down the length of the shortest side.

8

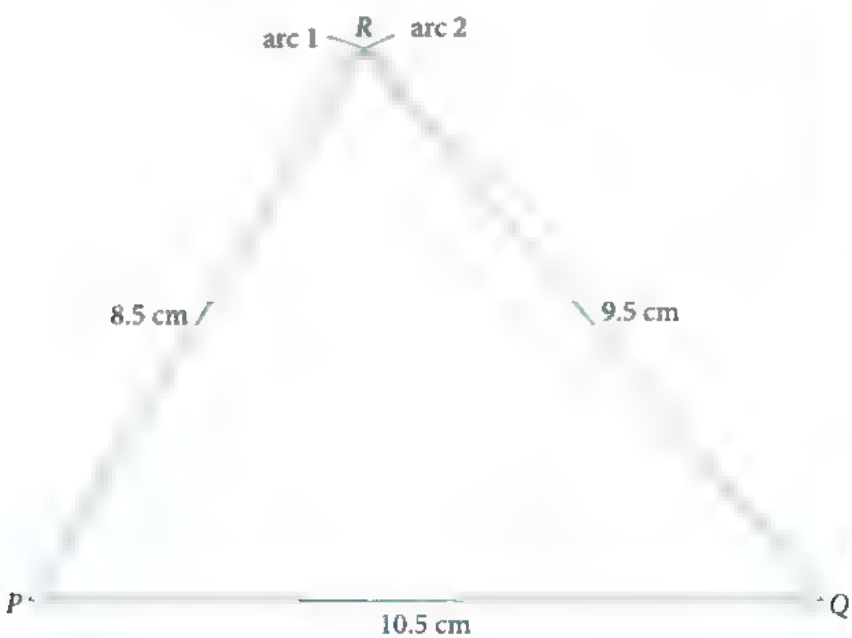
Constructing triangle given 3 sides

Construct $\triangle PQR$ such that $PQ = 10.5$ cm, $PR = 8.5$ cm and $QR = 9.5$ cm. Measure and write down the size of the angle facing the longest side.

Solution

Construction steps

1. Using a ruler, draw $PQ = 10.5$ cm.
2. Since R is 8.5 cm away from P, with P as centre and 8.5 cm as radius, draw arc 1.
3. Since R is 9.5 cm away from Q, with Q as centre and 9.5 cm as radius, draw arc 2 to cut arc 1 at R.
4. Join PR and QR.



The angle facing the longest side PQ is $\angle PRQ = 71°$.

Attention

Alternatively, the line PR or QR can be drawn first.
Do not erase the two arcs as they are construction lines.

Constructing geometrical diagrams helps us visualise spatial relationships to obtain new information from the given information, such as the size of the angle facing the longest side, as shown in this Worked Example.

Problem-solving Tip

We can read up to the smallest marking on a protractor. Hence, if the measurement is somewhere between 71° and 72°, decide if it is nearer to 71° or 72°.

Construct $\triangle PQR$ such that $PQ = 8.4$ cm, $PR = 7.2$ cm and $QR = 9.8$ cm. Measure and write down the size of the angle facing the longest side.

Exercise 11C

Worked Example 9

Constructing triangle given 2 sides and 1 angle

Construct $\triangle ABC$ such that $AB = 7.5$ cm, $BC = 5$ cm and $\angle BAC = 30°$. Measure and write down the possible lengths of AC.

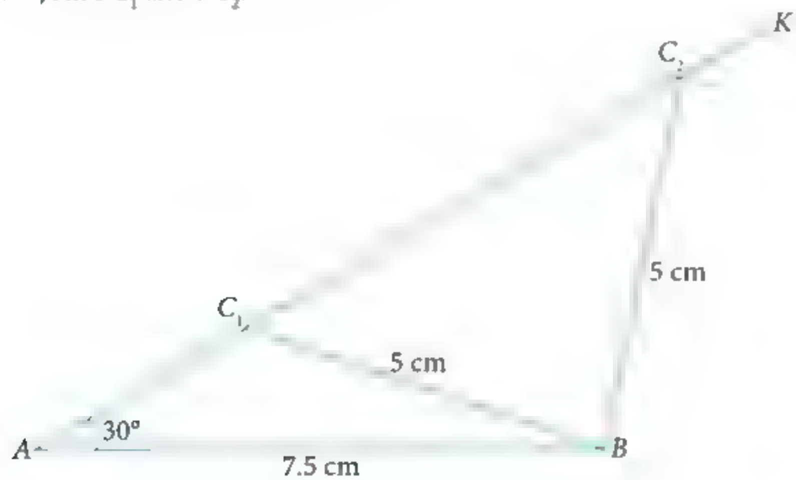
Solution

Construction steps

1. Using a ruler, draw $AB = 7.5$ cm.
2. Since $\angle A = 30°$, using a protractor at A, mark off an angle of 30° and draw a line AK such that $\angle BAK = 30°$.
3. Since C is 5 cm away from B, with B as centre and 5 cm as radius, draw an arc to cut AK at C_1, and another arc to cut AK at C_2.
4. Join BC_1 and BC_2.

Attention

- Sometimes, it is possible to construct two different triangles.
- If we draw the line BC first, we do not have any given angle to construct the line AB.



Possible lengths of AC are 3.2 cm and 9.8 cm.

1. Construct $\triangle ABC$ such that $AB = 9$ cm, $AC = 7$ cm and $\angle ABC = 45°$. Measure and write down the possible lengths of BC.
2. Construct $\triangle XYZ$ such that $XY = 7.6$ cm, $YZ = 4.8$ cm and $X\hat{Y}Z = 130°$. Measure and write down the length of XZ.

1. The steps for constructing the triangles in Worked Examples 7 – 9 are quite different. Why?
2. What do the construction steps depend on? How do I remember them?

C. Construction of quadrilaterals

In this section, we will learn how to construct quadrilaterals using a protractor and a pair of compasses and solve related problems.

10

Constructing parallelogram

Construct a parallelogram $ABCD$ such that $AB = 8.5$ cm, $BC = 6.5$ cm and $A\hat{B}C = 45°$. Measure and write down the length of the diagonal AC.

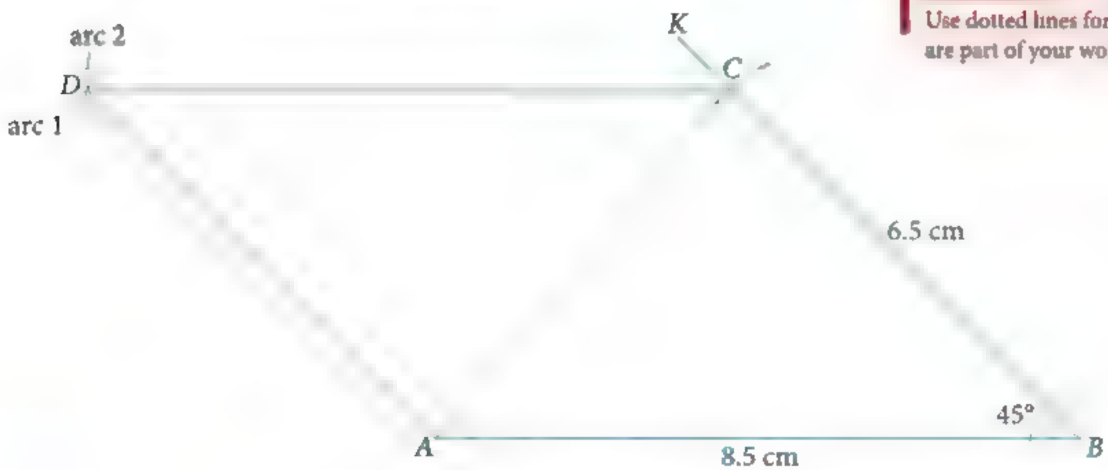
Solution

Construction steps:

1. Using a ruler, draw $AB = 8.5$ cm.
2. Since $\angle B = 45°$, using a protractor at B, mark off an angle of 45° and draw a line BK such that $A\hat{B}K = 45°$.
3. Since C is 6.5 cm away from B, with B as centre and 6.5 cm as radius, draw an arc to cut BK at C.
4. Since $ABCD$ is a parallelogram, $AD = BC = 6.5$ cm. With A as centre and 6.5 cm as radius, draw arc 1.
5. Similarly, $CD = BA = 8.5$ cm. With C as centre and 8.5 cm as radius, draw arc 2 to cut arc 1 at D.
6. Join AD and CD.
7. Join AC using a dotted line.

Attention

Use dotted lines for lines that are part of your working.



Length of $AC = 6.0$ cm

Exercise 11C

1. Construct a parallelogram $ABCD$ such that $AB = 8.5$ cm, $BC = 5.5$ cm and $A\hat{B}C = 120°$. Measure and write down the length of the diagonal AC.
2. Construct a rectangle $ABCD$ such that $AB = 10.5$ cm and $BC = 6.5$ cm. Measure and write down the length of the diagonal AC.

Worked Example 11

Constructing quadrilateral

Construct a quadrilateral $PQRS$ such that $PQ = 6.5$ cm, $QR = 4.8$ cm, $RS = 8.5$ cm, $P\hat{Q}R = 75°$ and $Q\hat{R}S = 98°$.

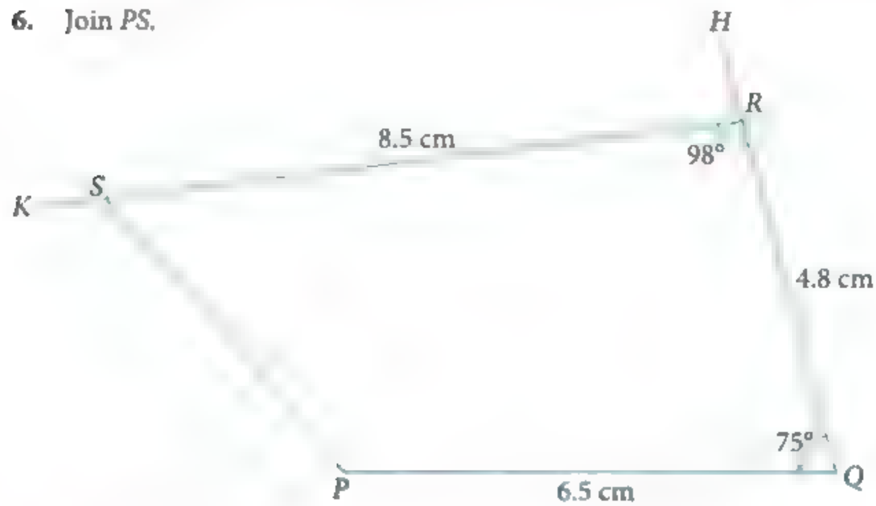
(i) Measure and write down the length of PS.

(ii) Measure and write down the size of $P\hat{S}R$.

Solution

Construction steps:

1. Using a ruler, draw $PQ = 6.5$ cm.
2. Since $\angle Q = 75°$, using a protractor at Q, mark off an angle of 75° and draw a line QH such that $P\hat{Q}H = 75°$.
3. Since R is 4.8 cm away from Q, with Q as centre and 4.8 cm as radius, draw an arc to cut QH at R.
4. Since $\angle R = 98°$, using a protractor at R, mark off an angle of 98° and draw a line RK such that $Q\hat{R}K = 98°$.
5. Since S is 8.5 cm away from R, with R as centre and 8.5 cm as radius, draw an arc to cut RK at S.
6. Join PS.



(i) Length of $PS = 4.8$ cm

(ii) $P\hat{S}R = 56°$

1. Construct a quadrilateral $PQRS$ such that $PQ = 5.6$ cm, $QR = 6.2$ cm, $RS = 9.2$ cm, $P\hat{Q}R = 80°$ and $Q\hat{R}S = 95°$.
   (i) Measure and write down the length of PS.
   (ii) Measure and write down the size of $P\hat{S}R$.
2. Construct a quadrilateral $PQRS$ such that $PQ = 6$ cm, $QR = 7.5$ cm, $RS = 8.2$ cm, $PS = 5.8$ cm and the diagonal $PR = 9.2$ cm. Measure and write down the size of $Q\hat{R}S$.

Exercise 11C

1. Construct $\triangle ABC$ such that $AB = 10.2$ cm, $A\hat{B}C = 60°$ and $B\hat{A}C = 45°$.
Measure and write down the length of AC.

2. Construct an isosceles triangle PQR such that $PQ = PR = 10$ cm and $QR = 9$ cm.
Measure and write down the size of $Q\hat{P}R$.

3. (a) Construct $\triangle ABC$ such that $AB = 8$ cm, $BC = 6.5$ cm and $A\hat{B}C = 80°$.
Measure and write down the length of AC.
(b) Construct $\triangle XYZ$ such that $XY = 5$ cm, $YZ = 9$ cm and $Y\hat{X}Z = 110°$.
Measure and write down the length of XZ.

4. Construct a parallelogram $ABCD$ such that $AB = 10$ cm, $BC = 12$ cm and $A\hat{B}C = 80°$. Measure and write down the length of the diagonal BD.

5. Construct a trapezium $WXYZ$ such that WZ is parallel to XY, $WX = 4.5$ cm, $XY = 8$ cm, $WZ = 6$ cm and $W\hat{X}Y = 60°$. Measure and write down the length of YZ and of WY.

6. Construct a quadrilateral $PQRS$ such that $PQ = 4$ cm, $QR = RS = 4.8$ cm, $PS = 3.6$ cm and $Q\hat{P}S = 90°$.
(i) Measure and write down the length of QS.
(ii) Measure and write down the size of $Q\hat{R}S$.

7. Construct a quadrilateral $PQRS$ such that $PQ = 10$ cm, $QR = 6$ cm, $PS = 3.5$ cm, $Q\hat{P}S = 60°$ and $P\hat{Q}R = 45°$.
(i) Measure and write down the length of SR.
(ii) Measure and write down the size of $P\hat{R}S$.

8. Construct $\triangle CDE$ such that $CD = 8.8$ cm, $CE = 9.2$ cm and $DE = 10.4$ cm.
By stating a property of triangles, name the smallest angle in $\triangle CDE$. Measure and write down the size of this angle.

9. Construct $\triangle ABC$ such that $AB = 8.4$ cm, $AC = 7.5$ cm and $A\hat{B}C = 50°$.
Measure and write down the possible lengths of BC.

10. Construct a rhombus $ABCD$ such that $AB = 60$ mm and the diagonal $AC = 9$ mm.
(i) State a property of rhombus used in order to complete this construction.
(ii) Measure and write down the size of $B\hat{A}D$.

11. Construct a quadrilateral $PQRS$ such that $PQ = PS = PR = 9$ cm, $QR = 12$ cm and $RS = 7.5$ cm.
Measure and write down the size of $Q\hat{P}S$.

12. Construct a quadrilateral $PQRS$ such that $PQ = 11$ cm, $PR = 12$ cm, $QS = 8.5$ cm, $P\hat{Q}R = 90°$, $Q\hat{P}S = 50°$ and $Q\hat{R}S$ is obtuse.
(i) Measure and write down the size of $Q\hat{R}S$.
(ii) Construct a line parallel to PR that passes through S to meet QR produced at U. Measure and write down the length of RU.

13. You are asked to construct a triangle of a side 7 cm and two angles 60° and 80°.
(i) How many different triangles can you construct?
(ii) Construct the triangle with 7 cm as its shortest side.

14. You are asked to construct a triangle of sides 1 cm, 3 cm and 5 cm. Explain if you are able to do so.

15. Construct $\triangle TUV$ such that $TU = 10.2$ cm, $UV = 5.3$ cm and $U\hat{T}V = 20°$.
(i) How many different triangles can you construct? Can you explain why using a property of triangles?
(ii) Write down the length of the longest side and the largest angle in each case.

11.4 Polygons

A. What are polygons?

What are polygons?

We have learnt about triangles and quadrilaterals in the previous sections.

Triangles and quadrilaterals are examples of polygons. Fig. 11.15 shows some other examples of polygons.

Fig. 11.15

A **polygon** is a plane figure that satisfies the following conditions:

(i) it consists of a number of points (called vertices) and an equal number of *straight* line segments (called sides) joining consecutive pairs of the points;

(ii) no three successive points are collinear (i.e. no three successive points lie on the same straight line).

1. Condition (i) implies that a polygon must have at least 3 sides and it must be *closed* (i.e. there are no gaps in its boundaries). For example, the shapes shown in Fig. 11.16 are *not* polygons. Why?

Fig. 11.16

2. The first two polygons in Fig. 11.15 are called **simple polygons** because their boundaries do not cross themselves, i.e. the line segments do not intersect one another. Are the third and fourth polygons in Fig. 11.15 simple polygons? Explain.
3. The first polygon in Fig. 11.15 is called a **convex polygon** because *all* of its interior angles are less than 180°. Why are the second and third polygons in Fig. 11.15 not convex polygons?

A polygon that is not convex is called a **concave polygon** (notice that the second polygon in Fig. 11.15 *caves in* at that vertex whose interior angle is more than 180°).

In this section, we will study only simple convex polygons. The term 'polygon' from this point onwards refers to 'simple convex polygon'.

Naming of polygons

Polygons are named after the number of sides that they have. Search the Internet for a video titled 'The Polygon Song' (not the one titled 'Polygon Song Video'). Listen to the song and write down the names of the following polygons.

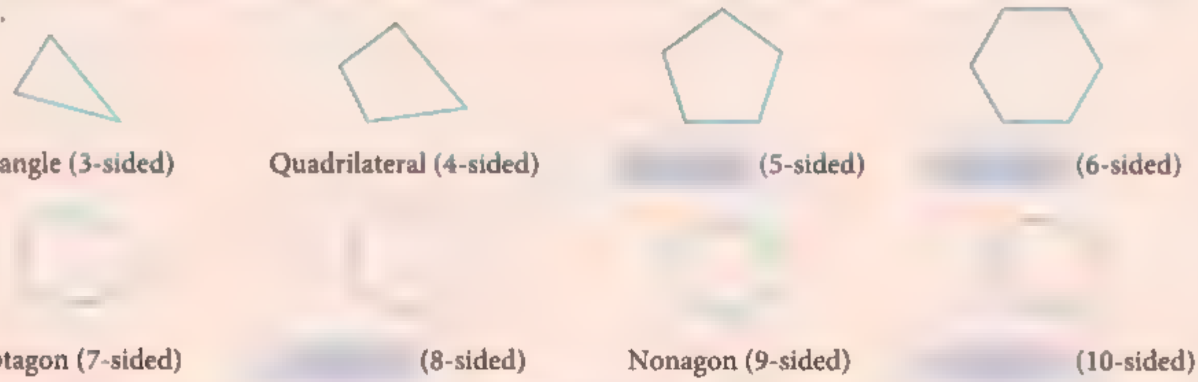


Fig. 11.17

The names of the polygons in Fig. 11.17 contain prefixes which are determined by their number of sides. Some polygons with more than 10 sides also have special names, but they are not easy to remember. We call a polygon with n sides an n-sided polygon or an **n-gon**. For example, a polygon with 12 sides is known as a 12-sided polygon or a 12-gon.

B. Regular polygons

A **regular polygon** is a polygon with *all sides equal* and *all interior angles equal*.

1. Which polygons in Fig. 11.17 in the above Investigation are regular polygons?
2. What is the name of a regular triangle and of a regular quadrilateral?

Definition of regular polygon

1. Is it possible for a polygon to have all sides equal without being a regular polygon?
   (a) What is the name of a non-regular quadrilateral with all sides equal?
   (b) Fig. 11.18 shows the pulling of a regular hexagon as indicated by the arrows to form a non-regular hexagon with all sides equal. Draw another non-regular hexagon with all sides equal but of a different shape as the one shown in Fig. 11.18.

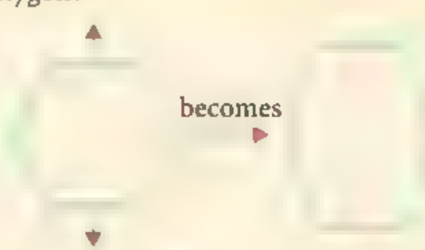


Fig. 11.18

2. Is it possible for a polygon to have all *interior* angles equal without being a regular polygon?
   (a) What is the name of a non-regular quadrilateral with all *interior* angles equal?
   (b) Draw two different non-regular hexagons with all *interior* angles equal.

C. Sum of interior angles of polygon

In Sections 11.1 and 11.2, we have learnt that the sum of the interior angles of a triangle and of a quadrilateral is 180° and 360° respectively. What is the sum of the interior angles of other polygons?

Sum of interior angles of polygon

In this Investigation, we will discover a general formula for the sum of interior angles of an n-sided polygon.

1. Copy and complete Table 11.4.

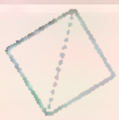
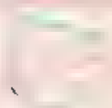
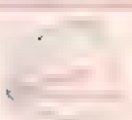
Polygon	Number of sides	Number of triangle(s) formed	Sum of interior angles
 Triangle	3	1	$1 \times 180° = (3-2) \times 180°$
 Quadrilateral	4	2	$2 \times 180° = (4-2) \times 180°$
Pentagon			
Hexagon			
 Heptagon			
 Octagon			
n-gon			

Table 11.4

2. From Table 11.4, what can you say about the number of triangles formed by a polygon in relation to the number of sides it has?
3. From Table 11.4, what is the general formula for the sum of interior angles of an n-sided polygon?
4. How can you tell that the general formula will always work for any n-sided polygon?

   Consider the pentagon in Fig. 11.19. What happens if you add a point (represented by the cross) to make it into a hexagon? Will you add one more triangle? What if the point is somewhere else? Will you always add one more triangle when you change a pentagon into a hexagon?

Pentagon

Fig. 11.19

From the Investigation on page 287, we observe that:

Sum of interior angles of polygon

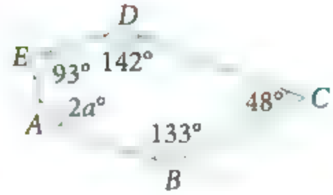
The sum of interior angles of an n-sided polygon is $(n - 2) \times 180°$.

Worked Example 12

Finding value of unknown in pentagon

(a) Find the sum of the interior angles in a pentagon.

(b) Hence, calculate the value of a in the figure.



Solution

(a) Sum of interior angles of an n-gon $= (n - 2) \times 180°$

Sum of interior angles of a pentagon $= (5 - 2) \times 180°$

$= 540°$

Problem-solving Tip

A pentagon has 5 sides, i.e. $n = 5$.

(b) $2a° + 133° + 48° + 142° + 93° = 540°$

$2a° = 540° - 133° - 48° - 142° - 93°$

$= 124°$

$a° = 62°$

$\therefore a = 62$

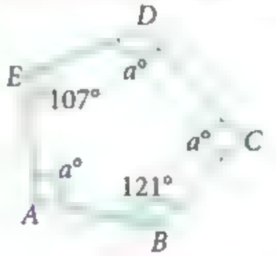
Practise Now 12

Similar and Further Questions

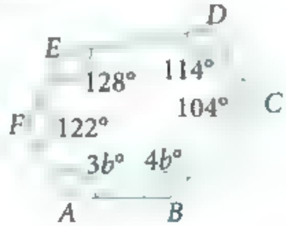
Exercise 11D Questions

1. Find the value of the unknown in each of the following figures.

(a)



(b)



2. Two of the interior angles of an n-sided polygon are 74° and 136°, and the remaining interior angles are 110° each. Find the value of n.

Worked Example 13

Interior angles of regular decagon

(i) Calculate the sum of interior angles of a regular decagon.

(ii) Hence, find the size of each interior angle of a regular decagon.

Problem-solving Tip

A decagon has 10 sides. In a regular decagon, all the interior angles are equal.

(i) Sum of interior angles of a regular decagon

$= (10 - 2) \times 180°$

$= 1440°$

(ii) Each interior angle of a regular decagon $= \dfrac{1440°}{10}$

$= 144°$

(i) Find the sum of interior angles of a regular polygon with 24 sides.
(ii) Hence, calculate the size of each interior angle of a regular polygon with 24 sides.

D. Sum of exterior angles of polygon

In Section 11.1, we have learnt that there are two ways to draw the 3 exterior angles of a triangle. Fig. 11.20 shows a pentagon *ABCDE* with *AB* produced to *P*, *BC* produced to *Q*, *CD* produced to *R*, *DE* produced to *S* and *EA* produced to *T*. $\angle p$, $\angle q$, $\angle r$, $\angle s$ and $\angle t$ are the **exterior angles** of the pentagon. Notice that the exterior angles go in an anti-clockwise direction from $\angle p$ to $\angle q$ to $\angle r$ to $\angle s$ to $\angle t$.

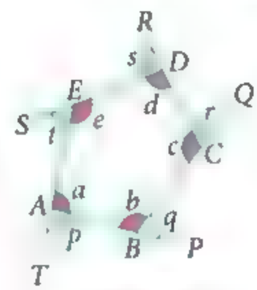


Fig. 11.20

Draw the exterior angles of the same pentagon in Fig. 11.20 in a clockwise direction.

Although there are two ways to draw the exterior angles of a pentagon, note that a pentagon has *exactly 5 exterior angles*, just like it has exactly 5 interior angles.

Sum of exterior angles of pentagon

Go to www.sl education.com/tmsoupp1/pg289 or scan the QR code on the right and open the geometry software template 'Exterior Angles of Polygon'. The template shows a pentagon with 5 exterior angles (see Fig. 11.21).

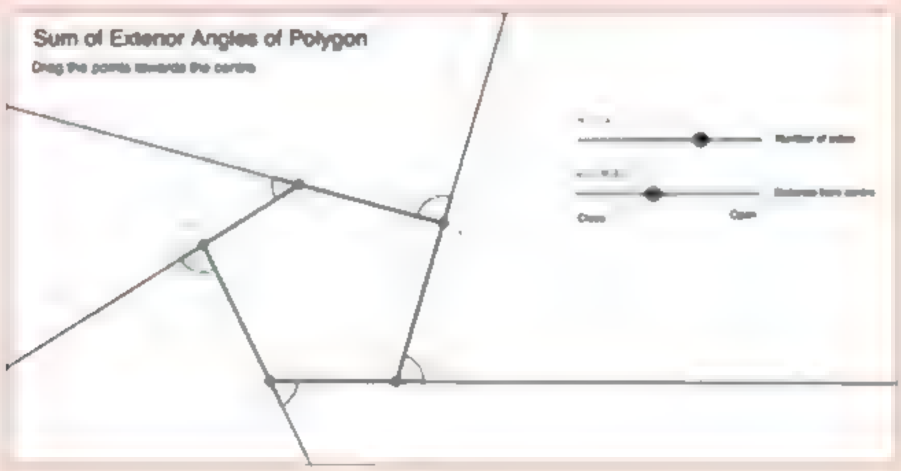


Fig. 11.21

1. Click on the slider labelled 'Distance from centre' to drag the points towards the centre. Fig. 11.22 shows the figure just before the points meet.

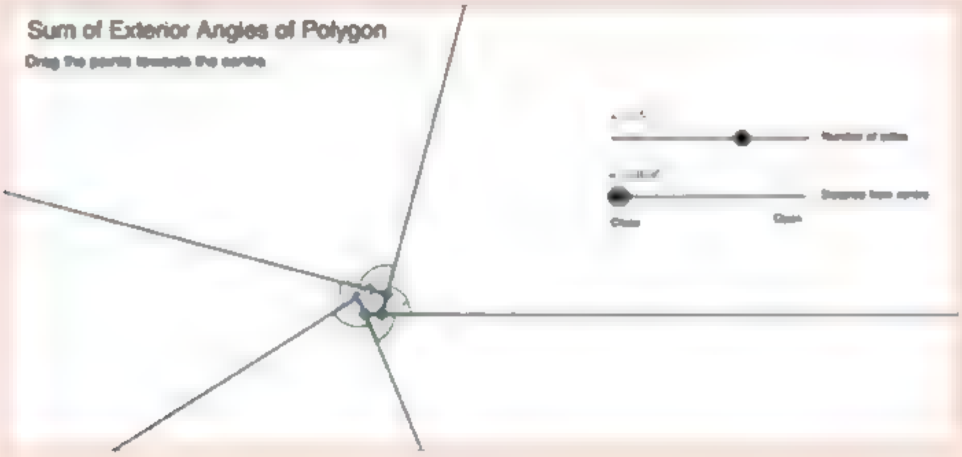


Fig. 11.22

2. What do you think the sum of the exterior angles of a pentagon is? Explain your answer.
3. Click on the slider labelled 'Number of sides' to look at other polygons. What do you think the sum of the exterior angles of an n-sided polygon is? Explain your answer.

Copy to complete the following proof that the sum of exterior angles of a pentagon is 360°.

Fig. 11.23

Consider the pentagon in Fig. 11.23.

Since $\angle a + \angle p = 180°$, $\angle b + \angle q = 180°$, $\angle c + \angle r =$ ____, $\angle d + \angle s =$ ____ and $\angle e + \angle t =$ ____, then

$$\angle a + \angle p + \angle b + \angle q + \angle c + \angle r + \angle d + \angle s + \angle e + \angle t = \text{____} \times 180°$$

$\therefore (\angle a + \angle b + \angle c + \angle d + \angle e) + (\angle p + \angle q + \angle r + \angle s + \angle t) = 900°$

Since the sum of interior angles of a pentagon $= \angle a + \angle b + \angle c + \angle d + \angle e$

$= (5 - 2) \times 180° = 540°$, then

$$540° + (\angle p + \angle q + \angle r + \angle s + \angle t) = 900°.$$

$$\therefore \angle p + \angle q + \angle r + \angle s + \angle t = 900° - \text{____}$$

$$= \text{____}$$

Similar and Further Questions
Exercise 11D
Question 14

Using a similar proof as shown in the Thinking Time on page 290, we can show that the sum of exterior angles of a hexagon, of a heptagon and of an octagon is also 360°. In general, we have:

Sum of exterior angles of polygon

The sum of exterior angles of any polygon is 360°.

In the Thinking Time on page 290, and the Investigation in Section 11.4C, no matter how you change the shape or size of each polygon, the properties for interior and exterior angles remain invariant.

Exterior angles of polygon

1. Is it possible for a regular polygon to have an exterior angle of 70°? Explain your answer.
2. If an exterior angle of a regular polygon is an integer, what are all the possible values of the angle?

Worked Example 14

Finding number of sides of regular polygon

Calculate the number of sides of a regular polygon if
(a) each exterior angle of the polygon is 24°,
(b) each interior angle of the polygon is 162°.

Solution

(a) The sum of exterior angles of the regular polygon is 360°.

$$\therefore \text{ number of sides of the polygon} = \frac{360^\circ}{24^\circ} = 15$$

(b) **Method 1:**

Let the number of sides of the regular polygon be n.

$$\text{Each interior angle of the polygon} = \frac{(n-2)\times 180^\circ}{n}$$

$$\frac{(n-2)\times 180^\circ}{n} = 162^\circ$$

$$(n-2)\times 180^\circ = 162^\circ \times n$$

$$180n - 162n = 360$$

$$18n = 360$$

$$n = 20$$

Method 2

Each exterior angle of the regular polygon $= 180^\circ - 162^\circ$
$= 18^\circ$ (adj. ∠s on a str. line)

$$\therefore \text{ number of sides of the polygon} = \frac{360^\circ}{18^\circ} = 20$$

Problem-solving Tip

For Method 2, we find each exterior angle using this relationship:
int. ∠ + ext. ∠ = 180° (adj. ∠s on a str. line).

Reflection

Which method do you prefer? Why?

Practise Now 14

Similar and Further Questions

Exercise 11D

Questions 4(a)–(d), 5(a)–(d)

1. Find the number of sides of a regular polygon if
   (a) each exterior angle of the polygon is 40°,
   (b) each interior angle of the polygon is 178°.
2. By finding the size of each exterior angle of a regular decagon, find the size of each interior angle of the decagon.
3. Two of the exterior angles of an n-sided polygon are 25° and 26°, three of its interior angles are 161° each and the remaining interior angles are 159° each. Calculate the value of n.

Worked Example 15

Problem involving regular pentagon

ABCDE is a regular pentagon. If *AB* and *DC* are produced to meet at *F*, find the value of $\angle BFC$.

We will use Polya's Problem Solving Model to guide us in solving this problem.

Stage 1: Understand the problem

We need to sketch a diagram to help us understand the problem.

What does the notation ABCDE tell us about the order of the vertices of the pentagon?

What is given? What are we supposed to find?

Diagrams help us visualise the given information so that we can think of a solution. Notations help to convey ideas in a concise and precise manner, e.g. pentagon *ABCDE* means that its vertices must be in this order: *A*, *B*, *C*, *D* and *E*. But it does not matter whether the order of the vertices is in the clockwise or anticlockwise direction.

Stage 2: Think of a plan

$\angle BFC$ lies in $\triangle BFC$.

What formula or property of a triangle can we use to calculate $\angle BFC$?

We can use $\angle$ sum of $\triangle$. This means we need to calculate $\angle CBF$ and $\angle BCF$ first.

Again, what formula or property can we use?

Notice that $\angle CBF$ and $\angle BCF$ are exterior angles of the pentagon.

Stage 3: Carry out the plan

Each exterior angle of the pentagon $= \dfrac{360°}{5} = 72°$

$\therefore \angle CBF = \angle BCF = 72°$

$\angle BFC = 180° - \angle CBF - \angle BCF$ ($\angle$ sum of $\triangle BCF$)

$= 180° - 72° - 72°$

$= 36°$

Stage 4: Look back

Is the answer reasonable?

Since we have sketched the diagram quite proportionally, $\angle BFC$ should be acute and smaller than $\angle CBF$ or $\angle BCF$. $\therefore$ 36° seems reasonable.

Are there other methods?

There are at least two other methods.

Hint for alternative method 1: $\angle ABC = \angle BCF + \angle BFC$ (ext. $\angle$ of $\triangle BCF$).

Hint for alternative method 2: $AEDF$ is a quadrilateral.

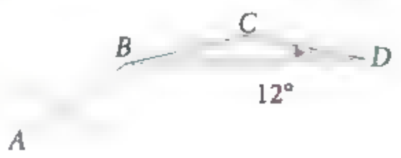
Exercise

$ABCDEF$ is a regular hexagon. If AB and DC are produced to meet at G, calculate $\angle BGC$.

Worked Example 16

Problem involving regular polygon with unknown number of sides

AB, BC and CD are adjacent sides of an n-sided regular polygon.



If $B\hat{D}C = 12°$, calculate

(i) the size of an exterior angle of the polygon,

(ii) the value of n,

(iii) $A\hat{B}D$.

Solution

(i) $C\hat{B}D = 12°$ (base $\angle$s of isos. $\triangle BCD$)

Size of each exterior angle of the polygon $= 12° + 12°$ (ext. $\angle$ of $\triangle BCD$)

$= 24°$

(ii) Sum of exterior angles of the n-sided regular polygon $= 360°$

$$\therefore n = \frac{360°}{24°} = 15$$

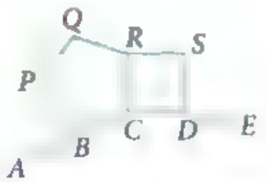
(iii) $A\hat{B}C + 24° = 180°$ (adj. $\angle$s on a str. line)

$$A\hat{B}C = 180° - 24° = 156°$$

$$A\hat{B}D = A\hat{B}C - C\hat{B}D = 156° - 12° = 144°$$

Similar and Further Questions
Exercise 11D
Questions 12, 13, 16

In the figure, $ABCDE$ is part of an n-sided regular polygon, $BPQRC$ is a regular pentagon and $CRSD$ is a square.



Calculate

(i) $P\hat{B}C$, **(ii)** $Q\hat{C}R$,
(iii) $B\hat{C}D$, **(iv)** $B\hat{D}C$,
(v) the value of n.

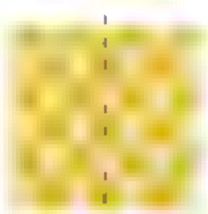
Introductory Problem Revisited

In the , you may not have known how to determine which other regular polygons can be used to tessellate the plane without any gaps, and why these polygons are able to form tessellations. After learning how to calculate the interior angle of a regular polygon, do you know how to answer the questions? Discuss these with your classmates.

1. Are there other ways to prove the formula for the sum of interior angles of a polygon?
2. Can each problem on polygons be solved using the formula for the sum of interior angles or the formula for the sum of exterior angles? Which method do I prefer?
3. What have I learnt in this section or chapter that I am still unclear of?

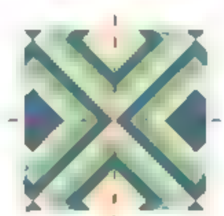
E. Line symmetry

A **line of symmetry** divides a plane figure into two identical halves such that if you fold the figure along it, each half of the figure will overlap with the other exactly. Fig. 11.24 shows examples of plane figures that exhibit line symmetry.



(a)

The dotted line shows the 1 line of symmetry.



(b)

The dotted lines show the 2 lines of symmetry.

Fig. 11.24

Line symmetry of special quadrilaterals and regular polygons

Refer to the 6 types of special quadrilaterals in Table 11.4 on page 287.
Fold paper cut-outs of these quadrilaterals to explore their line symmetries.

1. How many lines of symmetry does each of the quadrilaterals have?
2. From the line symmetry, are you able to observe any other geometrical relationships? For example, from the line of symmetry of a kite, which angles are equal?

Fold paper cut-outs of a regular pentagon, a regular hexagon and a regular octagon to explore their line symmetry.

3. How many lines of symmetry are there in each of the regular polygons?

F. Rotational symmetry

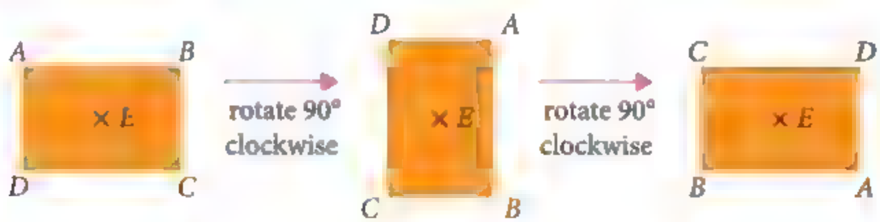
Let us learn about a new type of symmetry called rotational symmetry.
Fig. 11.25 shows examples of plane figures that exhibit rotational symmetry.

(a) The order of rotational symmetry of the rectangle is 2.

(b) The order of rotational symmetry of the snowflake is 6.

Fig. 11.25

The **order of rotational symmetry** is the number of distinct ways a plane figure maps onto itself in a rotation of 360° about its centre. For example, when we rotate a rectangle $ABCD$ about its centre E as shown in Fig. 11.26, $ABCD$ maps onto itself after a 180° rotation.



(a) **(b)** **(c)**

Fig. 11.26

In a full 360° rotation, $ABCD$ will map onto itself *twice*, so its order of rotational symmetry is 2.

If a plane figure maps onto itself only once in a 360° rotation about its centre, its order of rotational symmetry is 1. We also say that this figure has *no rotational symmetry*, since any irregular-shaped figure will also map onto itself once in a 360° rotation.

Rotational symmetry of special quadrilaterals and regular polygons

Refer to the 6 types of special quadrilaterals in Table 11.4 on page 287
Rotate paper cut outs of these quadrilaterals about their centre to explore each of their rotational symmetries.

1. Do the quadrilaterals have rotational symmetry? If yes, what is its order of rotational symmetry?
2. Why must the quadrilateral be rotated about its centre? What happens if it is rotated about any other point, such as about any of its vertices?
3. From the rotational symmetry, are you able to observe any other geometrical relationships? For example, since a parallelogram has a rotational symmetry of order 2, its opposite angles must be equal.

Rotate paper cut-outs of a regular pentagon, a regular hexagon and a regular octagon to explore each of their rotational symmetries.

4. Does each of the regular polygons have rotational symmetry? If so, what is its order of rotational symmetry?
5. From the rotational symmetry, are you able to observe any other geometrical relationships? For example, in the hexagon shown in Fig. 11.27, using rotational symmetry, we can see that there are 6 identical triangles.
   Can you identify other angles which are equal to $\angle x$?

Fig. 11.27

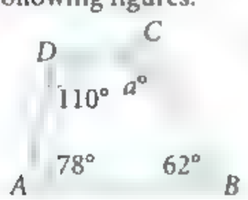
Basic | Intermediate

Exercise

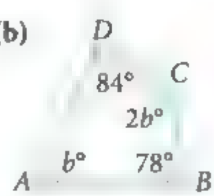
1. Find the sum of the interior angles of each of the following polygons.
   (a) 11-gon (b) 12-gon
   (c) 15-gon (d) 20-gon

2. Find the value of the unknown in each of the following figures.

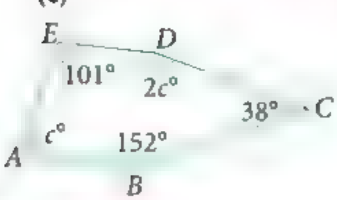
(a)



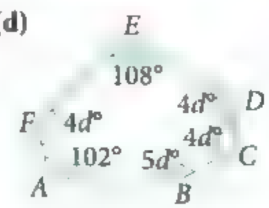
(b)



(c)



(d)



3. (a) (i) Find the sum of interior angles of a regular octagon.
   (ii) Hence, find the size of each interior angle of a regular octagon.
   (b) (i) Find the sum of interior angles of a regular polygon with 18 sides
   (ii) Hence, find the size of each interior angle of a regular polygon with 18 sides.

4. Find the number of sides of a regular polygon if each exterior angle of the polygon is
   (a) 45°, (b) 90°,
   (c) 4°, (d) 120°.

Exercise 11D

5. Find the number of sides of a regular polygon if each interior angle of the polygon is
(a) 140°, (b) 162°,
(c) 172°, (d) 175°.

6. (a) By finding the size of each exterior angle of a regular polygon with 24 sides, calculate the size of each interior angle of the polygon.
(b) By finding the size of each exterior angle of a regular polygon with 36 sides, calculate the size of each interior angle of the polygon.

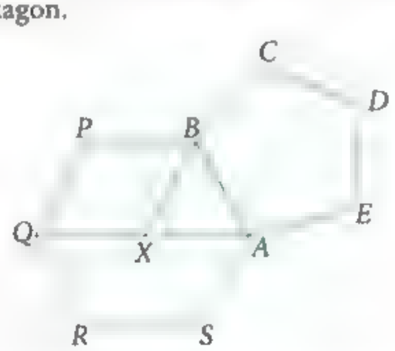
7. Three of the interior angles of an n-sided polygon are 76°, 169° and 105°, and the remaining interior angles are 146° each. Find the value of n.

8. Three of the exterior angles of an n-sided polygon are 50° each, two of its interior angles are 127° and 135°, and the remaining interior angles are 173° each. Find the value of n.

9. The ratio of an interior angle to an exterior angle of an n-sided regular polygon is 13 : 2. Find the value of n.

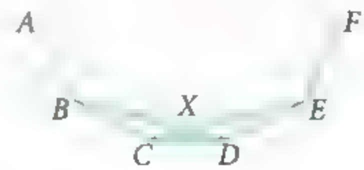
10. $ABCDEFG$ is a regular heptagon. If AB and DC are produced to meet at H, find the value of $\angle BHC$.

11. In the figure, $ABCDE$ is a regular pentagon and $ABPQRS$ is a regular hexagon. X is the centre of the hexagon.



Calculate
(i) $\angle ABP$, (ii) $\angle PQX$,
(iii) $\angle AXB$, (iv) $\angle ABC$,
(v) $\angle ACD$, (vi) $\angle ASE$.

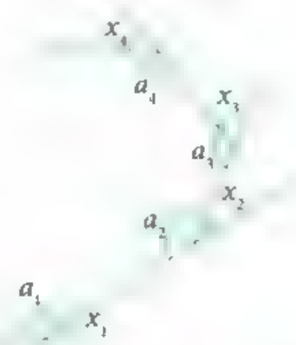
12. In the figure, $ABCDEF$ is part of an n-sided regular polygon. Each exterior angle of this polygon is 36°.



Calculate
(i) the value of n, (ii) $\angle BDE$,
(iii) $\angle BXE$.

13. The points A, B, C and D are consecutive vertices of a regular polygon with 20 sides. Calculate
(i) $\angle ABC$, (ii) $\angle ABD$.

14. The figure shows part of an n sided polygon where each side is produced. $a_1, a_2, a_3, a_4, \ldots$, and a_n are the interior angles of the polygon and $x_1, x_2, x_3, x_4, \ldots$, and x_n are its exterior angles.

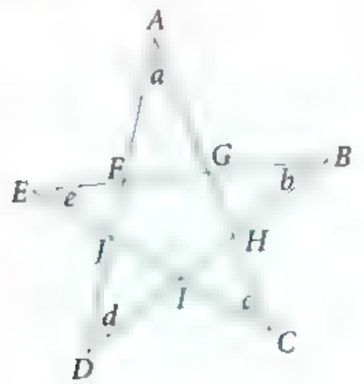


Show that the sum of exterior angles of the polygon is 360°, i.e. $x_1 + x_2 + x_3 + x_4 + \ldots + x_n = 360°$.

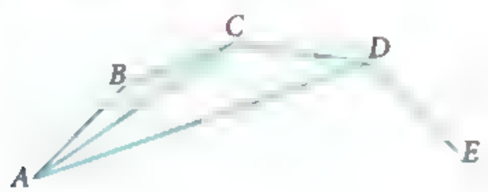
Basic | Intermediate

Exercise 11D

In the figure, $AFJD$, $AGHC$, $BGFE$, $BHID$ and $CIJE$ are straight lines. Find the value of $\angle a + \angle b + \angle c + \angle d + \angle e$. (Note: There are at least 3 methods to find the answer.)



In the figure, $ABCDE$ is part of an n-sided regular polygon. The ratio of an interior angle to an exterior angle of this polygon is 5 : 1.



Calculate

(i) the value of n, (ii) $\angle ACD$,

(iii) $\angle ADE$.

Hint: base angles of isosceles trapezium are equal

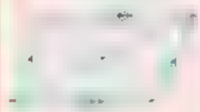
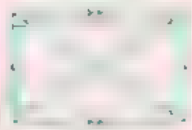
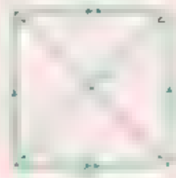
OPEN Q

A party is held in a large tent with a perimeter between 100 m and 150 m. A robot travels along the perimeter to serve refreshments to guests. It is programmed in such a way that it travels 3 metres, then turns through an angle of $x°$ clockwise, travels another 3 metres, then turns through an angle of $x°$ clockwise, and so on, until it reaches its initial position where the staff will top up the refreshments. Given that the robot is only able to turn through an integer value of x, find two possible values of x and the corresponding perimeters of the tent.

Polygons are everywhere in nature. For example, snowflakes or ice crystals are hexagonal in shape! As demonstrated by the Honeycomb Conjecture, angles offer a mathematical explanation for some natural occurences. In this chapter, we see that we can figure out the size of an interior angle and an exterior angle of a polygon using prior knowledge. Do you realise that new mathematics is often created based on what we already know? Every mathematical rule has a basis for it. One way to discover these new rules is to examine the properties that remained unchanged under certain changes. Remember how the formula for the sum of interior angles of an n-sided polygon is derived? We used the idea that the sum of interior angles of a triangle remains unchanged or **invariant** regardless of how the triangle is drawn. Invariant properties are important because they can be used to form new rules and solve problems in mathematics. Look back at Chapter 11, can you find other invariant properties?

1. **Properties of triangle**
   (a) Angle sum of triangle: The sum of interior angles of a triangle is 180° (∠ sum of △).
   (b) Exterior angle of triangle: An exterior angle of a triangle is equal to the sum of its interior opposite angles (ext. ∠ of △).

2. **Properties of quadrilaterals**
   (a) Angle sum of quadrilateral: The sum of the interior angles of a quadrilateral is 360° (∠ sum of quad.).
   (b) *Special quadrilaterals*

Special quadrilateral	**Parallelogram**	**Rectangle**	**Rhombus**	**Square**	**Kite**
		 (special type of parallelogram)	 (special type of parallelogram)	 (special type of rectangle and rhombus)	
Diagonals	Bisect each other	Bisect each other and are equal in length	Bisect each other at 90° and bisect the interior angles	Bisect each other at 90°, are equal in length and bisect the interior angles	Cut each other at 90° and one of them bisects the interior angles
Line(s) of symmetry	0	2	2	4	1
Order of rotational symmetry	2	2	2	4	1 (no rotational symmetry)

Some other inclusive definitions to note are:
- a parallelogram is a special type of trapezium,
- a rhombus is a special type of kite.

3. **Construction of triangle**
   A triangle can be constructed given
   (a) 1 side and 2 angles,
   (b) 3 sides,
   (c) 2 sides and 1 angle.
   - For (c), sometimes, it is possible to construct two different triangles.
   - Give an example of how a triangle can be constructed for each of the three cases above.

4. **Construction of quadrilateral**
   We make use of the properties of special quadrilaterals (e.g. parallelogram and rectangle) to construct them.
   - Give an example of how a kite can be constructed. What information do you need to construct the kite?

5. **Properties of polygons**
   (a) In a polygon, interior angle + exterior angle = 180°.
   (b) Sum of interior angles of an n-sided polygon = $(n - 2) \times 180°$.
   (c) Sum of exterior angles of an n-sided polygon = 360°.

CHAPTER 12

Perimeter and Area of Plane Figures

Plane figures such as triangles, circles, parallelograms and trapeziums are familiar to many of us. Designers often use these figures with special geometric properties to create interesting shapes. The iconic Toa Payoh dragon playground in Singapore, built in 1979, features a creature with a large dragon head and a body of colourful steel rings supported by trapezium-shaped pillars. When the playground was constructed, the builders likely needed to know the perimeter and area of each portion to figure out how much material was required.

Perimeter and area are two **measures** of the boundary of a plane figure: perimeter quantifies the length of the boundary while area quantifies the amount of space enclosed within the boundary.

In this chapter, we will learn how to find the perimeter and area of triangles, circles, parallelograms and trapeziums.

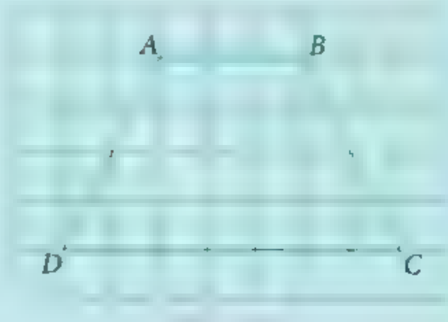
Learning Outcomes

What will we learn in this chapter?

- How to find the perimeter and area of plane figures (such as squares, rectangles, triangles, circles, parallelograms and trapeziums)
- How to solve problems involving the perimeter and area of plane figures (including composite figures)
- Why perimeter and area of plane figures have useful applications in real life

Introductory Problem

A trapezium is a four-sided figure with at least one pair of parallel sides. Fig. 12.1 shows a trapezium $ABCD$.



(a) Can you find the area of trapezium $ABCD$ by counting the number of square units?

(b) Use another method to find the area of trapezium $ABCD$.

(c) What can you learn from the different ways of finding the area of a trapezium?

(d) The trapezium given is a special one, where $AD = BC$. How can you use what you have learnt above to find the area of any given trapezium?

In this chapter, we will learn how to find the areas of parallelograms and trapeziums using what we have previously learnt about the area of rectangles. But first, we need to know how to convert cm^2 to m^2 and vice versa.

12.1 Conversion of units

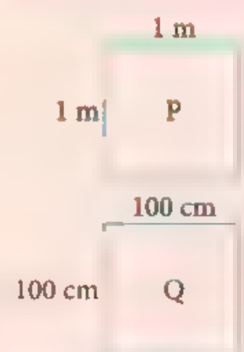
The floor area of a classroom is measured in square metres (m^2). Other units used to measure area of plane figures include square centimetres (cm^2), square millimetres (mm^2) and square kilometres (km^2).

Sometimes, we need to **convert** from one unit of area to another. For example, it is more common to say that the land area of Singapore is 720 km^2 instead of 720 000 000 m^2. Why?

Similarly, we do not usually say that the area of the cross section of a 10-cent coin is 0.000 269 m^2. Instead, we say that its cross-sectional area is 2.69 cm^2. Why?

Converting between cm^2 and m^2

The diagram shows 2 squares P and Q.



1. Are the two squares identical (i.e. exactly the same)? Why or why not?
2. If the two squares are identical, will their areas be equal?
3. Calculate the area of square P in m^2.
4. Calculate the area of square Q in cm^2.
5. Based on Questions 1 to 4, how many cm^2 are equal to 1 m^2? Explain.
6. If you were to cover square P with square tiles of length 1 cm, how many square tiles would you need? Why?

From the Investigation on page 302, we learn that $1\ m^2 = 10\ 000\ cm^2$.
Since $1\ m = 100\ cm$, why is $1\ m^2 \neq 100\ cm^2$?

Worked Example 1

Converting between cm^2 and m^2

Express

(a) $5\ m^2$ in cm^2,

(b) $200\ cm^2$ in m^2.

Solution

(a)

$$1\ m = 100\ cm$$
$$(1\ m)^2 = (100\ cm)^2$$
$$= 100\ cm \times 100\ cm$$
$$1\ m^2 = 10\ 000\ cm^2$$
$$5\ m^2 = 5 \times 10\ 000\ cm^2$$
$$= 50\ 000\ cm^2$$

(b)

$$100\ cm = 1\ m$$
$$1\ cm = \frac{1}{100}\ m$$
$$(1\ cm)^2 = \left(\frac{1}{100}\ m\right)^2$$
$$= \frac{1}{100}\ m \times \frac{1}{100}\ m$$
$$1\ cm^2 = \frac{1}{10\ 000}\ m^2$$
$$200\ cm^2 = 200 \times \frac{1}{10\ 000}\ m^2$$
$$= 0.02\ m^2$$

Problem-solving Tip

In **(a)**, you need to show how you obtain $1\ m^2 = 10\ 000\ cm^2$ from $1\ m = 100\ cm$ in your working.

Problem-solving Tip

In **(b)**, it will be easier to start with 100 cm on the left hand side (LHS) of the equal sign, unlike in **(a)** where we started with 1 m on the LHS.
We can also write $1\ cm = 0.01\ m$ and $1\ cm^2 = 0.0001\ m^2$.

Practise Now 1

Exercise 12A

Express

(a) $10\ m^2$ in cm^2,

(b) $22.5\ m^2$ in cm^2,

(c) $0.16\ m^2$ in cm^2,

(d) $300\ cm^2$ in m^2,

(e) $7146\ cm^2$ in m^2,

(f) $0.1\ cm^2$ in m^2.

12.2 Perimeter and area of rectangles and triangles

A. Perimeter and area of rectangles and squares (Recap)

Let us recall how to find the perimeter and area of a rectangle and a square.

Name	Figure	Perimeter	Area
Rectangle	l, b	$2(l + b)$	lb
Square	l, l	$4l$	l^2

Table 12.1

Measures
For a plane figure, the perimeter measures the length of the boundary of the figure while the area measures the space enclosed within the boundary of the figure. When is each of these measures useful?

2

Problem involving perimeter and area of rectangle

The length of a rectangular field is 4 m longer than its breadth

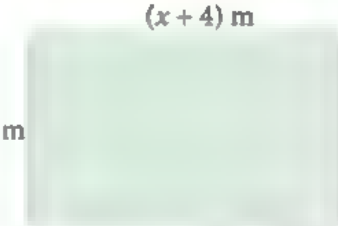
(a) If the perimeter of the field is 44 m, calculate
(i) the breadth, **(ii)** the area,
of the field.

(b) The field is surrounded by a cement path of width 2.5 m. Calculate the area of the path.

Solution

(a) **(i)** Let the breadth of the rectangular field = x m.

$(x + 4)$ m



x m

Then the length of the field = $(x + 4)$ m.

$$\therefore\ 2[(x + 4) + x] = 44$$
$$2(2x + 4) = 44$$
$$4x + 8 = 44$$
$$4x = 36$$
$$x = 9$$

$\therefore$ breadth of the field = 9 m

(ii) Area of the field = $(9 + 4) \times 9$
$= 117\ \text{m}^2$

(b) Total length = 13 + 2.5 + 2.5

= 18 m

Total breadth = 9 + 2.5 + 2.5

= 14 m

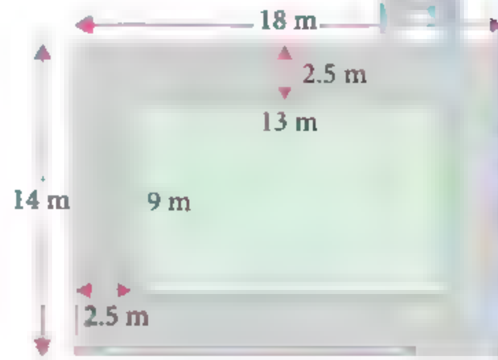
Total area of the field and the cement path

= 18 × 14

= 252 m^2

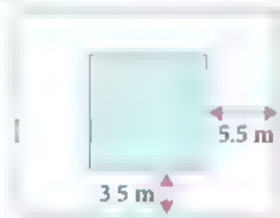
Area of the path = 252 – 117

= 135 m^2



Exercise 12A

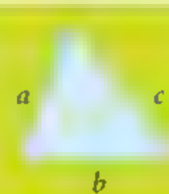
A rectangular park with a perimeter of 64 m has a square field in the centre and a running path surrounding the field. The diagram shows the park and its dimensions. Find the area of the running path.



B. Perimeter of triangles

The perimeter of a triangle is the sum of the lengths of its three sides. If a triangle has sides a, b and c, then

Perimeter of triangle = $a + b + c$



C. Base and height of triangle

To determine the area of a polygon, we must first identify the required dimensions of the polygon.

For a rectangle, the required dimensions are its length l, and breadth b. For a triangle, the required dimensions are its **height** h, and **base**, b.

Any side of a triangle can be its *base*, b. The *height*, h, of a triangle *with reference to the base* is the *perpendicular distance* from the base to the opposite vertex (see Fig. 12.2).

(a) (b) (c)

Fig. 12.2

3. Compare
   (a) the area of $\triangle ABC$ with rectangle $ABCD$,
   (b) the area of $\triangle JFG$ with rectangle $EFGH$.

   How are the areas of the triangles related to the areas of rectangles?

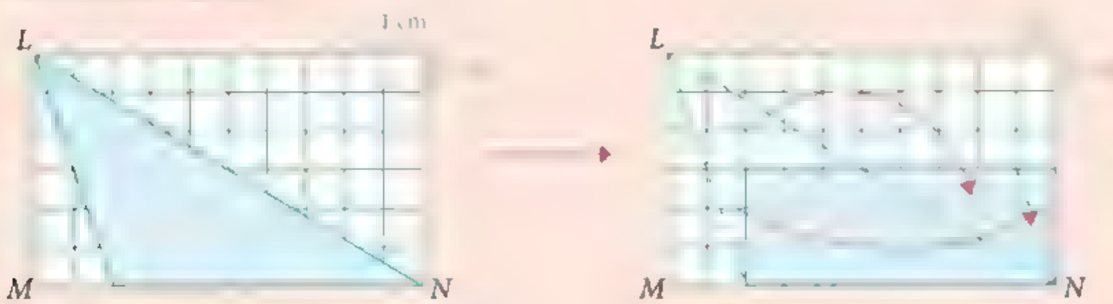
Part 2:

4. From Fig. 12.3 and Fig. 12.4, we have

Area of $\triangle ABC = \frac{1}{2} \times \underline{\qquad} \times BC$

Area of $\triangle JFG = \frac{1}{2} \times \underline{\qquad} \times FG$

5. Find the area of $\triangle LMN$.



Area of $\triangle LMN = \frac{1}{2} \times \underline{\qquad} \times \underline{\qquad}$

$= \underline{\qquad}$ cm^2

From the above Investigation, we have

Area of a triangle $= \frac{1}{2} \times$ base $\times$ height

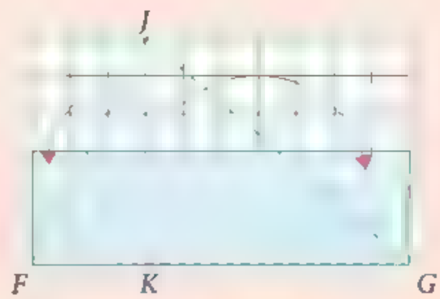
3. Compare
   (a) the area of $\triangle ABC$ with rectangle $ABCD$,
   (b) the area of $\triangle JFG$ with rectangle $EFGH$.

   How are the areas of the triangles related to the areas of rectangles?

Part 2:

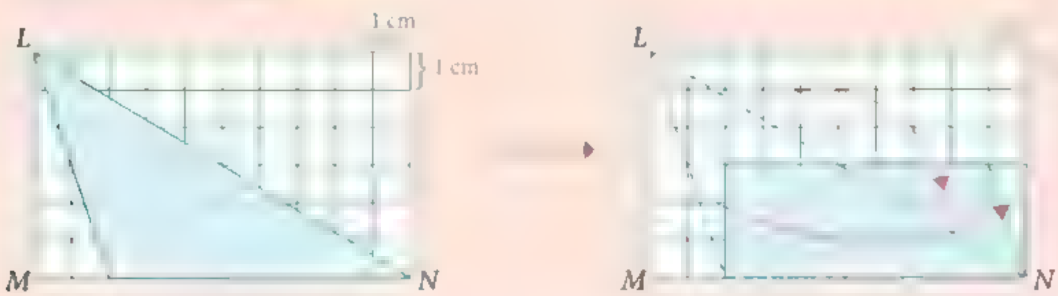
4. From Fig. 12.3 and Fig. 12.4, we have

Area of $\triangle ABC = \frac{1}{2} \times \square \times BC$



Area of $\triangle JFG = \frac{1}{2} \times \square \times FG$

5. Find the area of $\triangle LMN$.



Area of $\triangle LMN = \frac{1}{2} \times \square \times \square$

$= \square \text{ cm}^2$

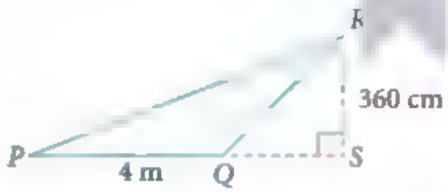
From the above Investigation, we have

Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

Worked Example 3

Finding area of triangle

The figure shows a triangle PQR such that $PQ = 4$ m and $PS = 360$ cm. Find the area of $\triangle PQR$.



Solution

Base = 4 m and height = 360 cm = 3.6 m　　recall 1 cm = (1 ÷ 100) m = 0.01 m

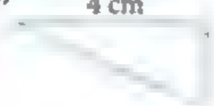
$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times 4 \times 3.6$$
$$= 2 \times 3.6$$
$$= 7.2\ \text{m}^2$$

Attention

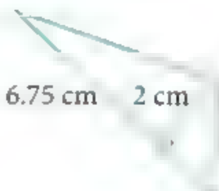
When finding area (or volume), convert all given lengths to the same unit.

Find the area of each of the following triangles.

(a)



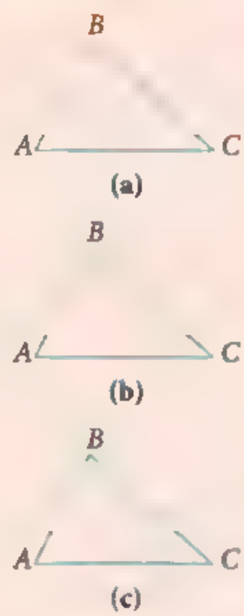
(b)



(c)

Area of triangle using different sides as base

How do we find the area of a triangle using different sides as the base? Let us investigate. Fig. 12.5 shows three identical $\triangle ABC$.



1. Taking AC as the base, draw and label the height, h_1, in Fig. 12.5(**a**).
2. Measure and record the length of AC and h_1.

   $AC =$ ___ cm　　　$h_1 =$ ___ cm
3. Calculate and record the area of $\triangle ABC$ using the formula given.

   Area of $\triangle ABC =$ ___ cm²
4. Repeat Steps 1 to 3.

   (i) Taking BC as the base in Fig. 12.5(**b**), find and label h_2.

   $BC =$ ___ cm　　　$h_2 =$ ___ cm　　　Area of $\triangle ABC =$ ___ cm²

   (ii) Taking AB as the base in Fig. 12.5(**c**), find and label h_3.

   $AB =$ ___ cm　　　$h_3 =$ ___ cm　　　Area of $\triangle ABC =$ ___ cm²

Is the area of $\triangle ABC$ obtained using the different pairs of heights and bases the same?

4

Finding height of triangle given area

The figure shows a triangle ABC such that $BC = 12$ cm. Given that the area of the triangle is 28 cm², find the height, h, of the triangle.

Solution

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$28 = \frac{1}{2} \times 12 \times h$$

$$= 6h$$

$$h = \frac{28}{6}$$

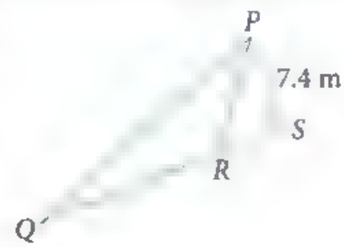
$$= 4.67 \text{ cm (to 3 s.f.)}$$

Attention

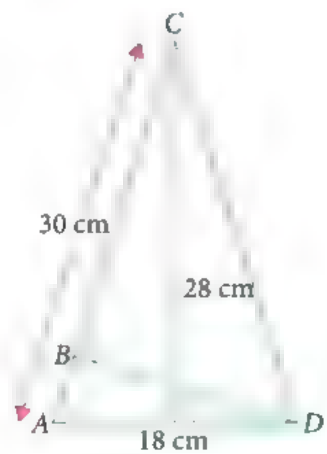
For measurements, we usually do not leave the answer as a fraction because the decimal form gives a clearer gauge of how long the object is.

Exercise 12A

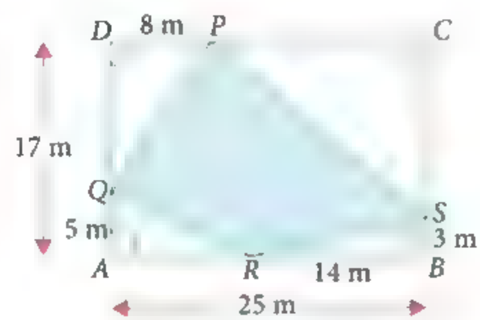
1. The figure shows a triangle PQR such that $PS = 7.4$ m. If the area of the triangle is 62.3 m², find the length of QR.



2. The diagram shows a triangle ACD such that $AD = 18$ cm and the height of the triangle is 28 cm.
   (i) Find the area of $\triangle ACD$.
   (ii) Given that BD is perpendicular to AC, find the length of BD.



3. In the figure, $AB = 25$ m, $AD = 17$ m, $DP = 8$ m, $AQ = 5$ m, $BR = 14$ m and $BS = 3$ m. Find the area of the shaded region.

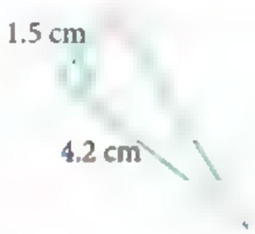


Exercise 12A

1. Express
   (a) 40 m^2 in cm^2, (b) 89.2 m^2 in cm^2,
   (c) 0.03 m^2 in cm^2, (d) 5.176 m^2 in cm^2.

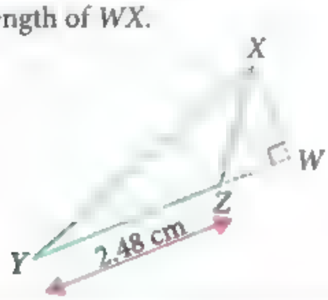
2. The area of a rectangle is 259 cm^2 and its length is 18.5 cm. Find
   (i) its breadth, (ii) its perimeter.

3. Find the area of each of the following triangles.
   (a)

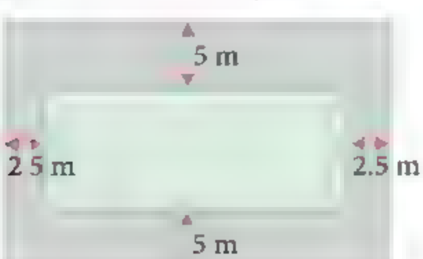


   (b)

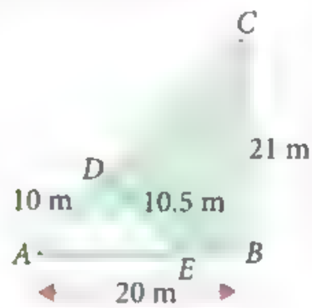
4. The figure shows a triangle XYZ such that $YZ = 2.48$ cm. If the area of the triangle is 2.31 cm^2, find the length of WX.



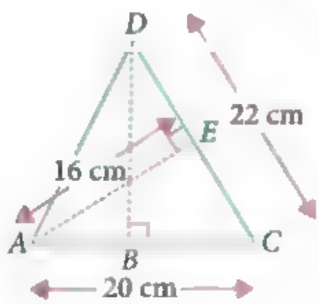
5. The perimeter of a rectangular field is 70 m and its length is 15 m longer than its breadth. The field is surrounded by a concrete path as shown in the figure. Find the area of the path.



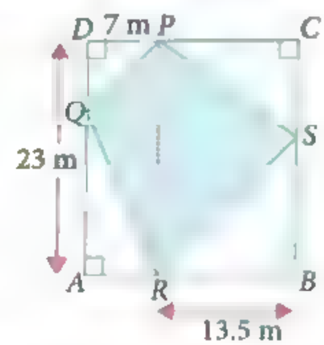
6. In the figure, $AB = 20$ m, $BC = 21$ m, $AD = 10$ m and $DE = 10.5$ m. Find the area of the shaded region.



7. In the diagram, $AC = 20$ cm, $CD = 22$ cm and $AE = 16$ cm. AE is perpendicular to CD and BD is perpendicular to AC. Find the length of BD.



8. In the figure, $AD = 23$ m, $DP = 7$ m and $BR = 13.5$ m.



   If P is directly above R, find the area of the shaded region.

OPEN

Write down the dimensions of three rectangles that have
(a) the same area but different perimeters,
(b) the same perimeter but different areas.

12.3 Perimeter and area of parallelograms

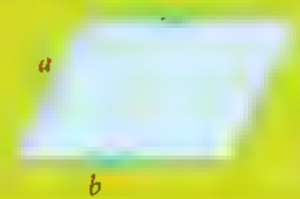
In primary school, we have learnt that a parallelogram is a 4-sided figure, with opposite sides that are parallel (and equal).

In this section, we will learn how to find the perimeter and the area of a parallelogram.

A. Perimeter of parallelogram

The perimeter of a parallelogram is the sum of the lengths of its four sides. Since each pair of opposite sides is of the same length in a parallelogram, then

Perimeter of parallelogram $= 2 \times$ (sum of lengths of adjacent sides) $= 2(a + b)$



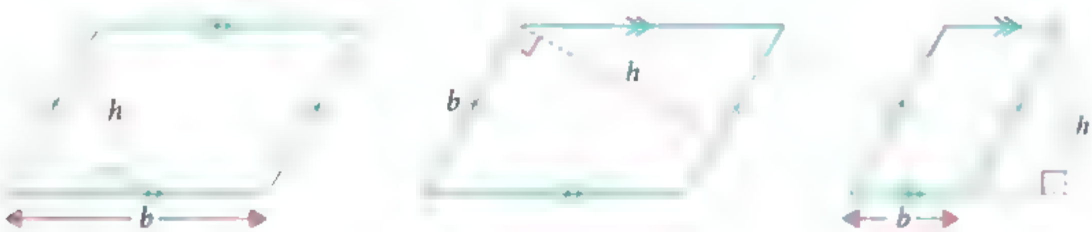
B. Base and height of parallelogram

To find the area of a parallelogram, we need to know its height.

The **height** of a parallelogram depends on which side of a parallelogram we use as its **base**.

We can use any side of a parallelogram as its base.

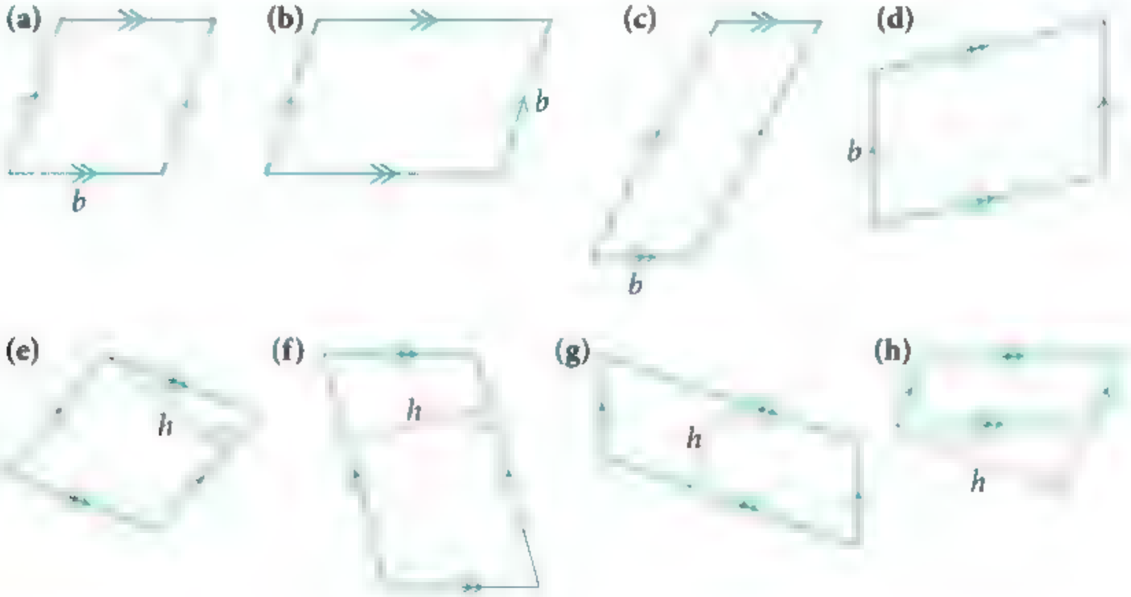
The height, h, of the parallelogram *with reference to the base* is the *perpendicular distance* from the base, b, to the opposite side (see Fig. 12.6).



(a) (b) (c)

Fig. 12.6

For each of the following parallelograms, label the height as h (or base as b) with reference to the given base (or height). You need to indicate the right angle where necessary.



C. Area of parallelogram

We have learnt from the Introductory Problem that we can cut a shape into two or more parts, and then rearrange these parts to form a shape that we know how to find the area of.

A parallelogram can be cut into two parts and rearranged into a rectangle.

Formula for area of parallelogram

In this Investigation, we will make use of the formula for the area of a rectangle to find a formula for the area of a parallelogram.

Fig. 12.7(**a**) shows a parallelogram $ABCD$ with base $AB = b$ and height $DE = h$.

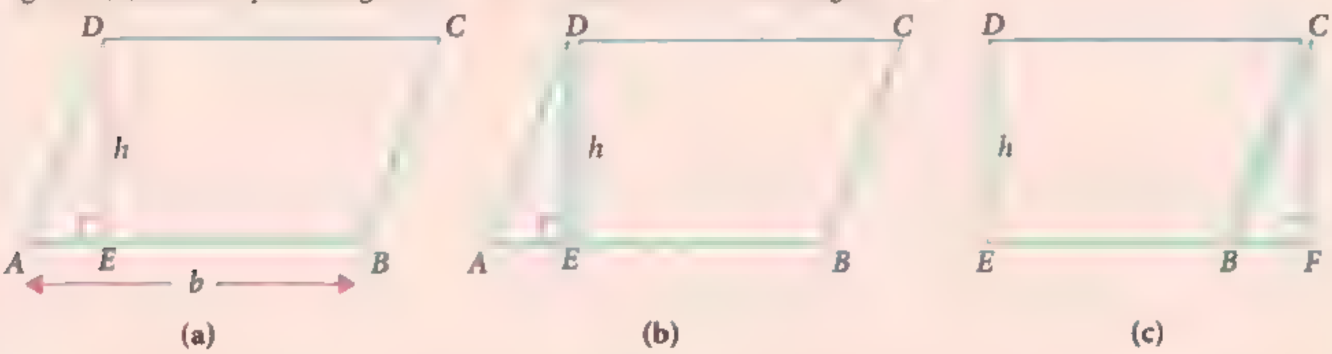


Fig. 12.7

1. If we remove the right-angled triangle ADE from the parallelogram $ABCD$ in Fig. 12.7(**b**) and place it as shown in Fig. 12.7(**c**), what is the shape of the new quadrilateral $CDEF$?
2. Find the length of CF and of EF in terms of b and h.
3. Hence, find a formula for the area of the parallelogram $ABCD$ in terms of b and h.
4. Think of another method to find a formula for the area of a parallelogram.

Divide the parallelogram $ABCD$ in another way and use the formula for the area of a triangle.

Fig. 12.8 shows a parallelogram that is slanted so far to one side such that we cannot draw the height inside the parallelogram and cut it as in the Investigation on page 312.

Does the formula which you have found in the Investigation for the area of a parallelogram still work for this oblique parallelogram? You may go to www.sl-education.com/tmsoupp1/pg313 or scan the QR code on the right and make use of the geometry software template 'Area of Parallelogram' to help you visualise. In your journal, explain why the formula works or does not work.

Fig. 12.8

From the Investigation on page 312 and the above Journal Writing, we observe that:

Area of parallelogram = base × height = bh

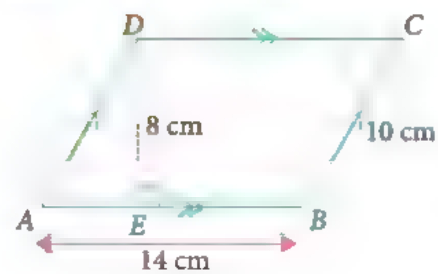
5

Finding perimeter and area of parallelogram

The diagram shows a parallelogram $ABCD$ where AB = 14 cm and BC = 10 cm. If DE = 8 cm, find

(i) the perimeter, (ii) the area,

of the parallelogram.



Solution

(i) Since the opposite sides of a parallelogram are equal in length, $AB = DC$ and $BC = AD$.

Perimeter of parallelogram = 2(14 + 10)

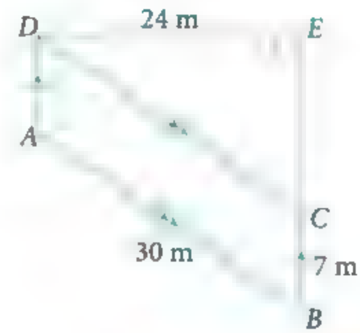
= 48 cm

(ii) Area of parallelogram = base × height

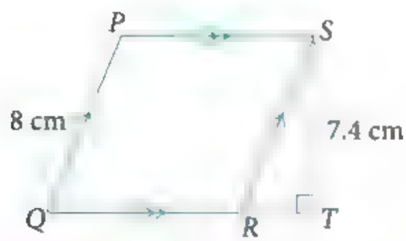
= 14 × 8

= 112 cm²

1. The diagram shows a parallelogram $ABCD$ where $AB = 30$ m and $BC = 7$ m. If $DE = 24$ m, find
   (i) the perimeter,
   (ii) the area,
   of the parallelogram.

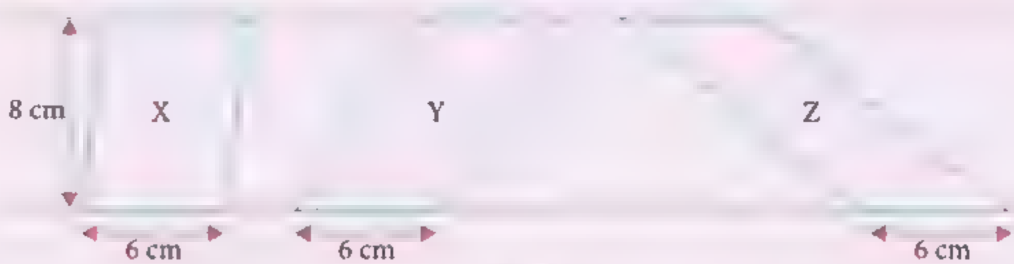


2. The diagram shows a rhombus $PQRS$ of length 8 cm. If $ST = 7.4$ cm, find
   (i) the perimeter,
   (ii) the area,
   of the rhombus.



In Chapter 11, we have learnt that a rhombus is a special parallelogram with four equal sides.

Fig. 12.9 shows three parallelograms X, Y and Z with base 6 cm and height 8 cm.



(i) Find the area of each of the parallelograms.
(ii) What do you notice about their areas? Why?

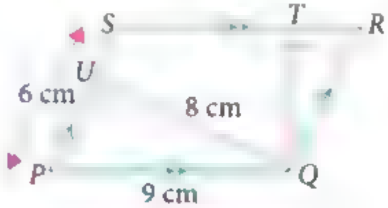
From the above Thinking Time, we observe that the areas of the three parallelograms are all equal because the area of a parallelogram depends only on its base and height. We say that the areas of all parallelograms with the same base and height are **invariant**, regardless of how slanted they are.

Invariance
As shown in the above Thinking Time, the area of a parallelogram is invariant when it is slanted parallel to its base, while keeping the base constant.

Worked Example 6

Finding unknown in parallelogram

The figure shows a parallelogram $PQRS$ where $PQ = 9$ cm and $PS = 6$ cm. QU is perpendicular to PS and QT is perpendicular to SR. If $QU = 8$ cm, calculate the length of QT, leaving your answer to one decimal place.



Solution

Area of parallelogram = base × height

$= PS \times QU$

$= 6 \times 8$

$= 48\text{ cm}^2$

Since $PQ \times QT$ = area of parallelogram, then

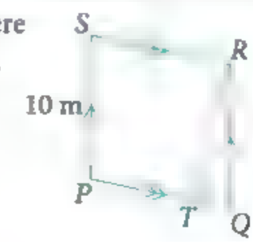
$$PQ \times QT = 48$$
$$9 \times QT = 48$$
$$QT = \frac{48}{9}$$
$$= 5.3\text{ cm (to 1 d.p.)}$$

Problem-solving Tip

To find the area of the parallelogram, we have to identify a base and its corresponding height that we know the lengths of. Since the length of QT is not given, we cannot use QT (height) and PQ (base) to find the area. Instead, we use PS (base) and QU (height).

Exercise 12B

The figure shows a rhombus $PQRS$ where $PS = 10$ m. If the area of triangle PRS is 48 m^2, find the length of RT.



Reflection

There is more than one method to solve this problem. Can you find the length of RT using two different methods?

7

Problem involving area of parallelogram

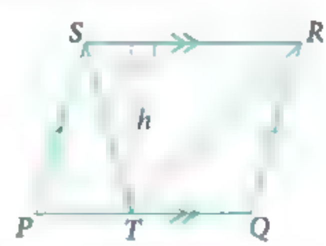
$PQRS$ is a parallelogram and T lies on PQ. Given that the area of triangle SRT is 30 cm^2, find the area of parallelogram $PQRS$. What do you notice about the areas of triangle SRT and parallelogram $PQRS$? Explain.

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: **Understand the problem**

We need to sketch a diagram to help us understand the problem. The notation $PQRS$ means that the vertices of the parallelogram must be in this order: P, Q, R and S. Since the question only says that T lies on PQ, then point T can be anywhere on PQ.

What other information is given? What are we supposed to find?



Stage 2: Think of a plan

Since only the area of a triangle is given, how can we calculate the lengths of the base and height of the parallelogram?

Observe that the triangle and parallelogram have the same base and height.

Attention

It does not matter whether the order of the vertices is in the clockwise or anticlockwise direction, or which vertex we label as P.

Stage 3: Carry out the plan

Let the perpendicular distance from T to SR be h cm.

$$\text{Area of triangle } SRT = \frac{1}{2} \times SR \times h = 30$$

$$SR \times h = 60$$

$$\text{Area of parallelogram } PQRS = SR \times h$$

$$= 60 \text{ cm}^2$$

The area of triangle SRT is half the area of parallelogram $PQRS$ because both shapes have the same base b and the same height h, but area of triangle $= \frac{1}{2}bh$ while area of parallelogram $= bh$.

Stage 4: Look back

Is the answer reasonable? Is there another way to explain why the area of the triangle is half the area of the parallelogram?

Hint: Draw a line from T to SR such that the line is parallel to PS.

The diagram shows a parallelogram $ABCD$ and a smaller parallelogram $PQRS$. P and S lie on AD, and Q and R lie on BC. Given that $AD : PS = 5 : 1$, find the ratio of the area of the parallelogram $ABCD$ to that of the parallelogram $PQRS$.

Worked Example 8

Perimeter and area of composite figures in real-world contexts

Regulatory signs with a chevron pattern, i.e. the V-shaped stripe, are commonly found on roads to regulate the movement of traffic. The chevron is made up of parallelograms and represents an extended curve of the road.

A diagram of the chevron is shown below, where $ABCD$ is a rectangle and ST is parallel to AB.

By dividing the chevron into equal parallelograms, find the percentage of this sign board that is painted yellow.

*Solution

Percentage of sign board that is painted yellow $= \frac{6}{12} \times 100\%$

$= 50\%$

Exercise 12B
Question 10

Simple geometrical shapes appear on some national flags too. For example, the flag of the Republic of Trinidad and Tobago is made up of two triangles and three parallelograms.

The width of each white portion is $\frac{1}{30}$ of the length of the flag and the width of the black portion is $\frac{2}{15}$ of the length of the flag. Calculate the following ratio:

area of red portion : area of black portion : area of white portion.

1. What do I already know about the height of a triangle that could guide my learning of the height of a parallelogram in this section?
2. What do I already know about the formula for the area of a rectangle that could guide my learning of the formula for the area of a parallelogram in this section?

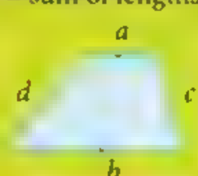
12.4 Perimeter and area of trapeziums

In Chapter 11, we have learnt that a trapezium is a 4-sided figure with at least one pair of parallel sides. In this section, we will learn how to find the perimeter and the area of a trapezium.

A. Perimeter of trapezium

Since the perimeter of a trapezium is the sum of the lengths of its four sides, then

Perimeter of trapezium = sum of lengths of all 4 sides = $a + b + c + d$



From the Investigation on page 318, we observe that:

$$\text{Area of trapezium} = \frac{1}{2} \times (\text{sum of lengths of parallel sides}) \times \text{height}$$
$$= \frac{1}{2}(a + b)h, \text{ where } a \text{ and } b \text{ are the lengths of the parallel sides}$$

How are the formulae for the area of a trapezium, a parallelogram and a triangle related to one another?

1. (i) If the parallel sides of a trapezium are equal in length (i.e. $a = b$), what is the shape of the new figure?
   (ii) If we substitute $a = b$ into the formula for the area of a trapezium, what do we get after simplification?
2. (i) If we reduce the length of one of the parallel sides of a trapezium until it becomes a point (i.e. $a = 0$), what is the shape of the new figure?
   (ii) If we substitute $a = 0$ into the formula for the area of a trapezium, what do we get after simplification?

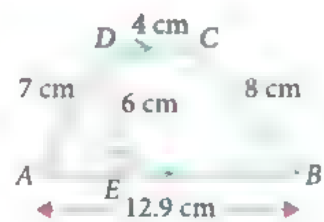
Worked Example 9

Finding perimeter and area of trapezium

The figure shows a trapezium $ABCD$ where $AB = 12.9$ cm, $BC = 8$ cm, $CD = 4$ cm and $AD = 7$ cm. If $DE = 6$ cm, calculate

(i) the perimeter, (ii) the area,

of the trapezium.



Solution

(i) Perimeter of the trapezium $= 12.9 + 8 + 4 + 7$
$= 31.9$ cm

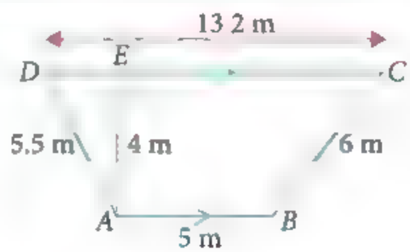
(ii) Area of the trapezium $= \frac{1}{2} \times (\text{sum of lengths of parallel sides}) \times \text{height}$
$= \frac{1}{2} \times (12.9 + 4) \times 6$
$= 50.7 \text{ cm}^2$

Practice Now 9

The figure shows a trapezium $ABCD$ where $AB = 5$ m, $BC = 6$ m, $CD = 13.2$ m and $AD = 5.5$ m. If $AE = 4$ m, find

(i) the perimeter, (ii) the area,

of the trapezium.

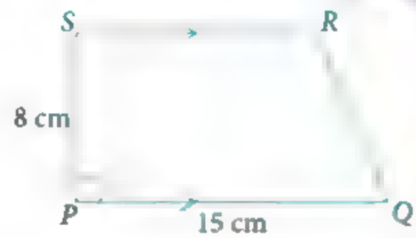


10

Finding unknowns in trapezium

The figure shows a trapezium $PQRS$ where $PQ = 15$ cm and $PS = 8$ cm. If the area and the perimeter of the trapezium are 104 cm^2 and 42.9 cm respectively, calculate the length of

(i) RS, (ii) QR.



Solution

(i) The height of the trapezium is given by the length of $PS = 8$ cm.

Area of the trapezium $= \frac{1}{2} \times$ (sum of lengths of parallel sides) $\times$ height

$$104 = \frac{1}{2} \times (15 + RS) \times 8$$

$$26 = 15 + RS$$

$$RS = 11$$

$\therefore$ length of $RS = 11$ cm

(ii) Perimeter of the trapezium $= PQ + QR + RS + PS$

$$42.9 = 15 + QR + 11 + 8$$

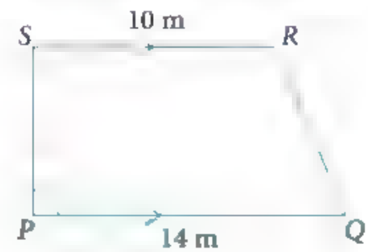
$$QR = 8.9$$

$\therefore$ length of $QR = 8.9$ cm

Exercise 12B

The figure shows a trapezium $PQRS$ where $PQ = 14$ m and $RS = 10$ m. If the area and the perimeter of the trapezium are 72 m^2 and 37.2 m respectively, find the length of

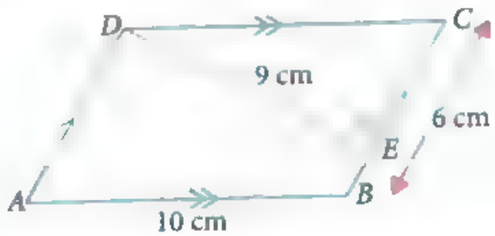
(i) PS, (ii) QR.



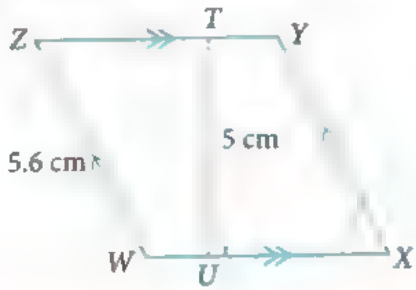
1. What do I already know about the height of a parallelogram that could guide my learning of the height of a trapezium in this section?
2. What do I already know about the formula for the area of a parallelogram that could guide my learning of the formula for the area of a trapezium in this section?

Exercise 12B

1 The figure shows a parallelogram $ABCD$ where $AB = 10$ cm and $BC = 6$ cm. If $DE = 9$ cm, find
(i) the perimeter, **(ii)** the area,
of the parallelogram.



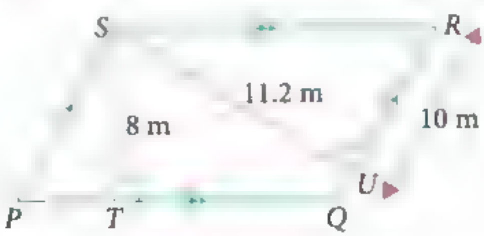
2 The figure shows a rhombus $WXYZ$ of length 5.6 cm. If $TU = 5$ cm, find
(i) the perimeter, **(ii)** the area,
of the rhombus.



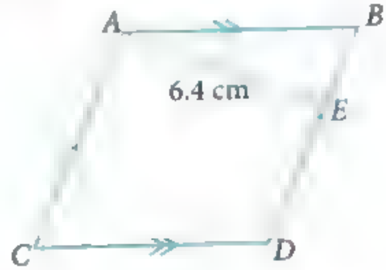
3 Complete the table for each parallelogram.

	Base	Height	Area
(a)	12 cm	7 cm	
(b)		6 m	42 m^2
(c)	7.8 mm		42.9 mm^2

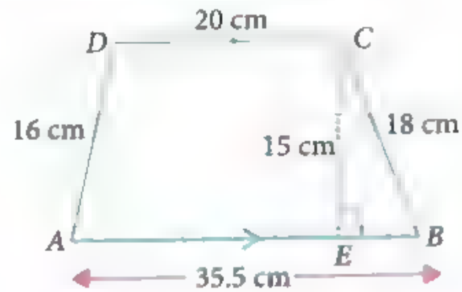
4 The figure shows a parallelogram $PQRS$ where $QR = 10$ m. If $ST = 8$ m and $SU = 11.2$ m, find the length of PQ.



5 The figure shows a rhombus $ABCD$ where $AE = 6.4$ cm. If the area of the rhombus is 44.8 cm^2, find the perimeter of the rhombus.



6 The figure shows a trapezium $ABCD$ where $AB = 35.5$ cm, $BC = 18$ cm, $CD = 20$ cm and $AD = 16$ cm. If $CE = 15$ cm, find
(i) the perimeter, **(ii)** the area,
of the trapezium.

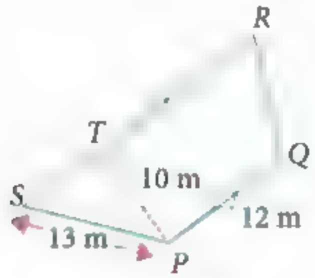


7 Complete the table for each trapezium.

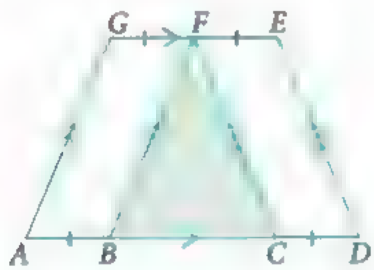
	Parallel side 1	Parallel side 2	Height	Area
(a)	7 cm	11 cm	6 cm	
(b)	8 m	10 m		126 m^2
(c)	5 mm		8 mm	72 mm^2

Exercise 12B

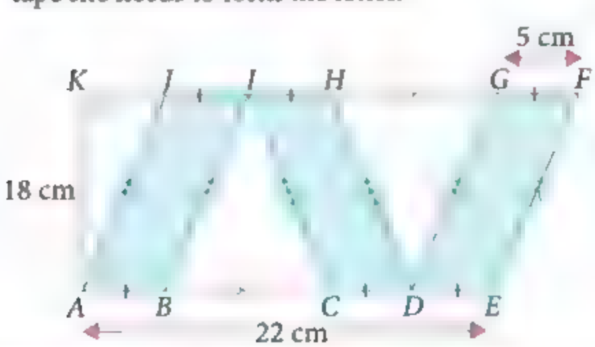
8. The figure shows a trapezium $PQRS$ where $PQ = 12$ m and $PS = 13$ m. If $PT = 10$ m, and the area and the perimeter of the trapezium are 185 m² and 61 m respectively, find the length of
   (i) RS, (ii) QR.



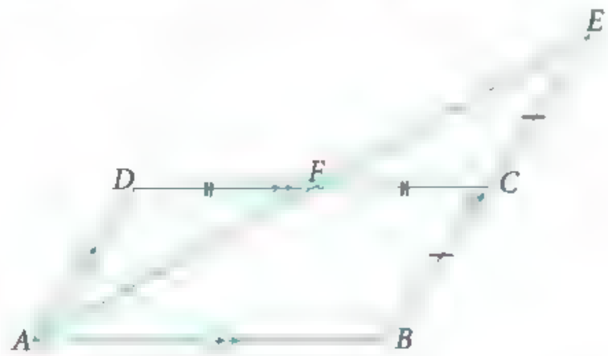
9. In the figure, $ABFG$ and $CDEF$ are two parallelograms such that the sum of their areas is 702 cm². If $AB = CD = EF = FG = \frac{1}{2}BC$, find the area of the shaded region.



10. Nadia wants to stick duct tape on her suitcase to form the letter 'N' for easy identification. The diagram shows her design for the letter 'N', where $AE = 22$ cm, $AK = 18$ cm and $AB = CD = DE = FG = HI = IJ = 5$ cm. Calculate the total area of the duct tape she needs to form the letter.



11. In the figure, $ABCD$ is a parallelogram, and AFE and BCE are straight lines. If the area of the parallelogram is 80 cm², $BC = CE$ and $DF = FC$, find the area of
   (i) $\triangle ABE$, (ii) $\triangle ADF$.



12. In the figure, $ABCD$ is a parallelogram and AED is a right-angled triangle. If the area of $\triangle AED$ is 25 cm², and the lengths of AE and EB are in the ratio 1 : 3, find the area of the trapezium $BCDE$.

13. On a sheet of graph paper, draw
   (a) two parallelograms of different dimensions but with the same area of 10 cm²,
   (b) two trapeziums of different dimensions but with the same area of 10 cm².

12.5 Circumference and area of circles

In Chapter 11, we learnt how to use a pair of compasses to draw a circle. In this section we will learn how to find the circumference and area of a circle.

A. Centre, radius and diameter of circle

When a pair of compasses is used to draw a circle as shown in Fig 12.12(a), the pointed end that is fixed forms the **centre**, *O*, of the circle.

A line segment drawn from *O* to a point *B* on the boundary of the circle is the **radius** (plural: *radii*) of the circle (see Fig. 12.12(**a**)).

The **diameter** of the circle *AB* is formed by extending *OB* to meet another point *A* on the boundary of the circle (see Fig. 12.12(**b**)).

How is the diameter of a circle, d, related to its radius, r?

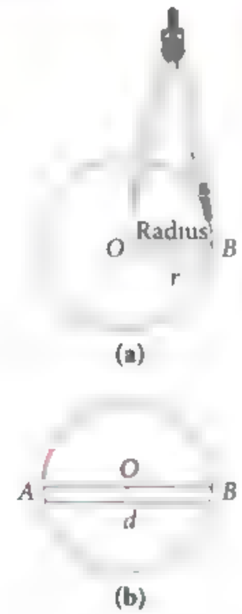


Fig. 12.12

In the figure, *O* is the centre of the circle. *A*, *B*, *C*, *D*, *E* and *F* are points located on the boundary of the circle.

(i) Name the diameters of the circle.

(ii) Name the radii of the circle.

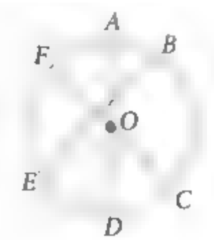
(iii) Compare the lengths of the line segments using the symbols '>', '=' or '<'.

(a) OA ____ OF

(b) AD ____ CF

(c) BE ____ CF

Provide a reason for your answer.

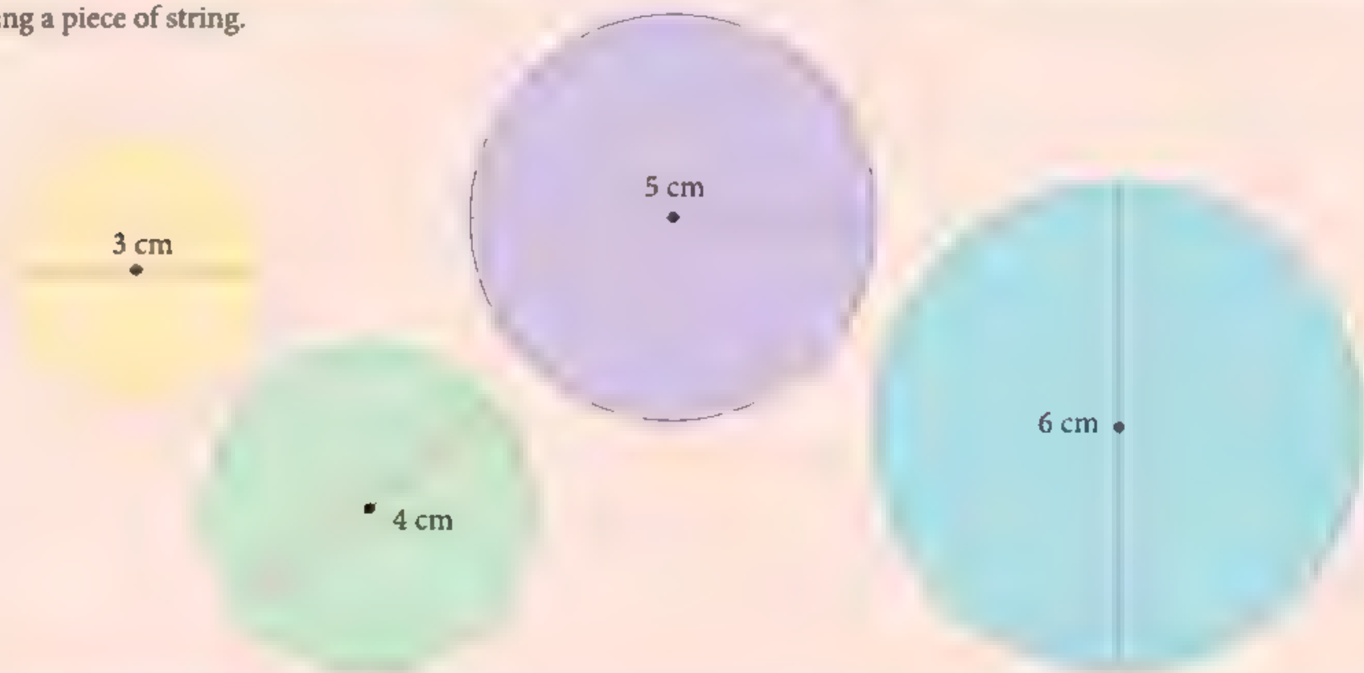


B. Circumference of circle

We have learnt that the perimeter of a closed 2D figure is the length of the boundary of the figure. For a circle, the length of the boundary is the **circumference**.

Formula for circumference of circle

The four circles shown are drawn to scale and their diameters are given. Measure the circumference of each circle using a piece of string.



1. Copy and complete Table 12.2.

Diameter	Circumference	$\dfrac{\text{Circumference}}{\text{Diameter}}$ (correct to 1 decimal place)
3 cm	9.4 cm	3.1
4 cm	12.5 cm	
5 cm	cm	
6 cm	cm	

Table 12.2

2. From Table 12.2, how many times of its diameter is the circumference of a circle?

From Table 12.2, the ratio $\dfrac{\text{Circumference}}{\text{Diameter}}$ has the same value for any circle.

This ratio is called **pi** and is represented by the symbol π.

That is,

Circumference of a circle
$= \pi \times$ diameter of circle
$= \pi d = 2\pi r$

Attention

In Chapter 4, we have learnt that π is an irrational number, i.e. it cannot be expressed as a ratio of two integers. π is sometimes approximated to 3.14 or $\frac{22}{7}$.

On a calculator, the value of π can be found by pressing

SHIFT

C. Area of circle

We have learnt from the Introductory Problem that we can cut a shape into two or more parts, and then rearrange these parts to form a shape that we know how to find the area of. To find a formula for the area of a circle, we can cut a circle into equal parts and rearrange the pieces.

Formula for area of circle

Fig. 12.13(**a**) shows a circle with a radius r that is divided into 24 equal parts. The pieces are rearranged to form a shape in Fig. 12.13(**b**).

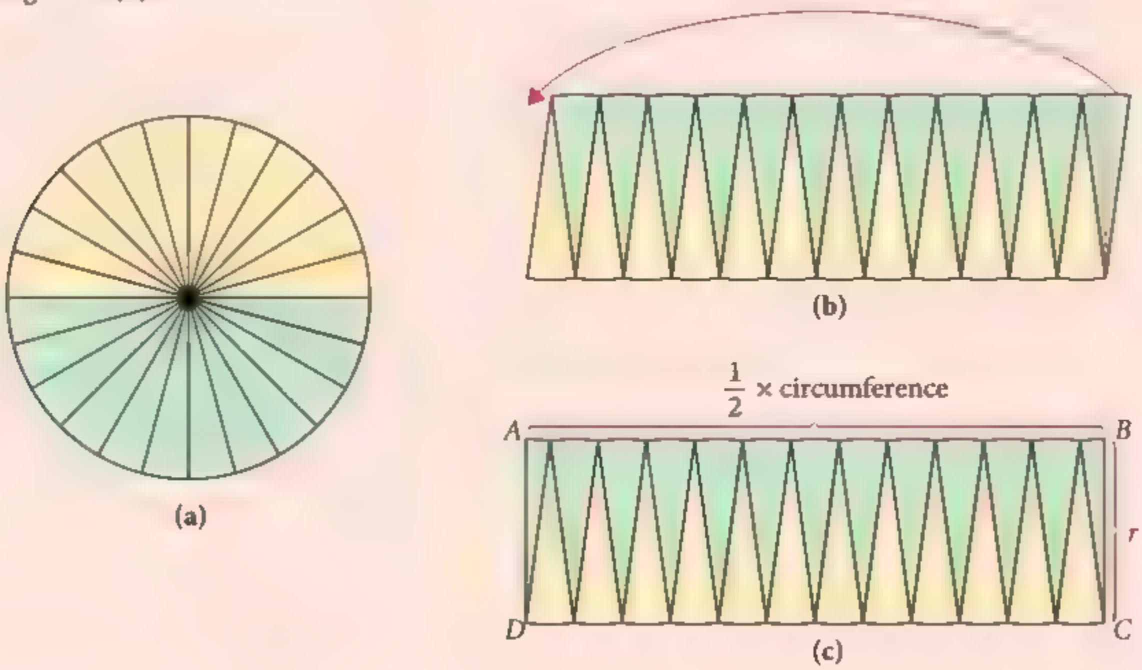


Fig. 12.13

1. If we cut the last piece into two equal halves and move one half to form $ABCD$ as shown in Fig. 12.13(**c**), which quadrilateral does $ABCD$ resemble?
2. Find the length AB in terms of r.
3. As the number of pieces increases, the shape of $ABCD$ approaches that specified in Question 1. Hence, find a formula for the area of the circle, which is equivalent to the area of the quadrilateral in Fig. 12.13(**c**), in terms of r.

From the above Investigation,

Area of circle = π × radius × radius = πr^2

Worked Example 11

Finding perimeter and area of circle

The figure shows a circle with centre O and a radius of 4.5 cm.

Calculate

(i) the circumference,

(ii) the area,

of the circle.

Solution

(i) Circumference of circle $= 2\pi r$

$$= 2 \times \pi \times 4.5$$

$$= 28.3 \text{ cm (to 3 s.f.)}$$

(ii) Area of circle $= \pi r^2$

$$= \pi \times (4.5)^2$$

$$= 63.6 \text{ cm}^2 \text{ (to 3 s.f.)}$$

1. Determine the circumference and area of a circle with a diameter of 12 cm.
2. The figure shows a semicircle with centre O and diameter AOB. If the radius of the circle is 8 cm, calculate the area and perimeter of the semicircle.

Worked Example 12

Finding unknown in circle

Determine the radius and area of a circle with a circumference of 64.5 m.

Solution

Circumference of circle = 64.5 m

$$2\pi r = 64.5$$

$$r = \frac{64.5}{2\pi}$$

$$= 10.3 \text{ m (to 3 s.f.)}$$

$\therefore$ radius of circle = 10.3 m

Area of circle $= \pi r^2$

$$= \pi \times \left(\frac{64.5}{2\pi}\right)^2$$

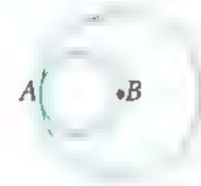
$$= 331 \text{ m}^2 \text{ (to 3 s.f.)}$$

Problem-solving Tip

For accuracy, we should use the intermediate value of $\frac{64.5}{2\pi}$ when calculating the area of the circle.

Exercise 12C

1. Determine the circumference of a circle that has an area of 804 cm^2.
2. The figure consists of a circle with diameter AB in a larger circle with centre B. If the circumference of the larger circle is 75 m, calculate
   (i) the length of AB,
   (ii) the circumference of the smaller circle, and
   (iii) the area of the smaller circle.



13

Problem involving circumference and area of circle

The figure shows a circle of radius 7 cm, touching two sides of a rectangle. The length of the rectangle is 9 cm longer than its width. Calculate
(i) the circumference of the circle,
(ii) the area of the circle,
(iii) the area of the shaded region.

Solution

(i) Circumference of circle $= 2\pi r$

$$= 2 \times \pi \times 7$$
$$= 44.0 \text{ cm (to 3 s.f.)}$$

(ii) Area of circle $= \pi r^2$

$$= \pi \times 7^2$$
$$= 49\pi$$
$$= 154 \text{ cm}^2 \text{ (to 3 s.f.)}$$

(iii) Width of rectangle = diameter of circle $= 7 \times 2$

$$= 14 \text{ cm}$$

Length of rectangle $= 14 + 9$

$$= 23 \text{ cm}$$

Area of shaded region = area of rectangle – area of circle

$$= 23 \times 14 - 49\pi$$
$$= 168 \text{ cm}^2 \text{ (to 3 s.f.)}$$

Attention

If the question does not specify the value of π, we use the value of π stored in the calculator.

The figure shows a circle of radius 14 cm with one quadrant removed, touching the sides of a square. Find
(i) the perimeter of the unshaded region,
(ii) the area of the unshaded region,
(iii) the area of the shaded region.

Work in pairs.

1. The dimensions of a rectangular living room of a 4-room flat are 6.6 m by 3.6 m. How many square tiles of length 60 cm does the owner need to tile the floor of the living room?
2. (a) In a particular apartment, the 4-room flats are between 85 m^2 and 105 m^2 in size. Fig. 12.14 shows a floor plan of a flat with the dimensions in mm. Can you find the floor area of this flat? Is it between 85 m^2 and 105 m^2?

   (b) The floor plan in Fig. 12.14 gives the dimensions in mm. Why do we calculate the floor area in m^2 instead of mm^2? How does this relate to the units of measure used for area (e.g. cm^2, m^2)?

Fig. 12.14

3. We often encounter real-world situations of a composite figure formed by two or more basic geometrical shapes. We can decompose a composite figure into basic geometrical shapes such as circles, triangles, rectangles, squares, parallelograms, or trapeziums, to find its perimeter and area. Which basic geometrical shape is the most useful? Why?

Exercise 12C

1. What do I already know about the formula for the area of a rectangle that could guide my learning of the formula for the area of a circle in this section?
2. What have I learnt in this section or chapter that I am still unclear of?

Intermediate Basic

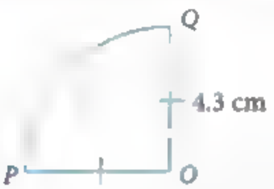
Exercise 12C

If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures.

1. Complete the table for each circle.

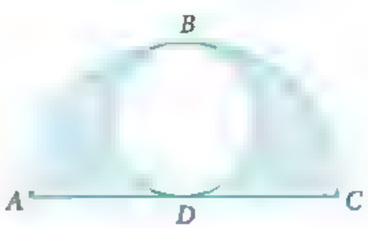
	Radius	Circumference	Area
(a)	8 cm		
(b)		34.6 cm	
(c)			302 mm²

2. The figure shows a quarter of a circle *POQ*. If $OP = OQ = 4.3$ cm, calculate



(i) the area, **(ii)** the perimeter,
of *POQ*.

3. The figure shows a circle with diameter *BD* encased in a semicircle *ABCD*.



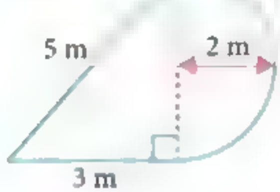
If the circumference of the smaller circle is 31.4 cm, determine

(i) the radius of the semicircle,

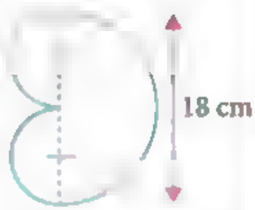
(ii) the area of the shaded region.

4. The circular portions of the following figures are semicircles. Find the perimeter and area of each of the following figures.

(a)

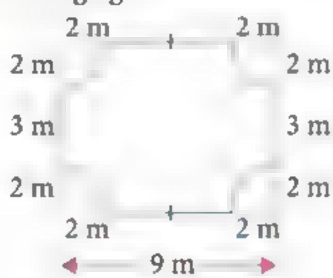


(b)



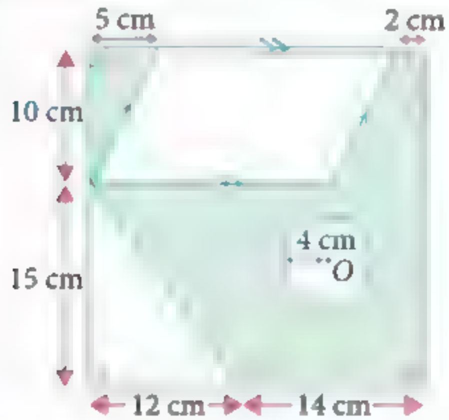
5. In the figure, 4 quadrants, each of radius 2 m, are removed from a rectangle. Find

(i) the perimeter, **(ii)** the area,
of the remaining figure.

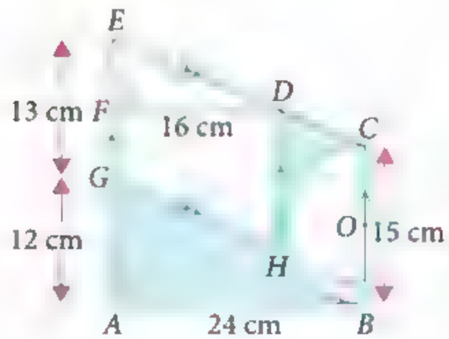


Exercise 12C

6. Find the total area of the shaded regions in the figure below, where O is the centre of the circle.



7. Find the area of the shaded region, where O is the centre of the circle.



8. The cross section of a foot stool and its dimensions are shown in Fig. (a):

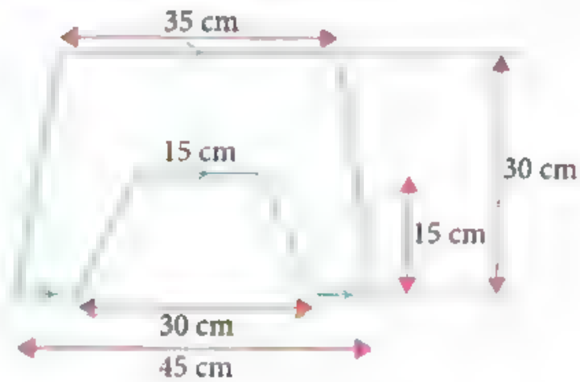


Fig. (a)

(i) Find the area of the cross section of the stool.

(ii) The manufacturer is considering changing the cross section of this stool to that shown in Fig. (b), where arc ABC is a semicircle.

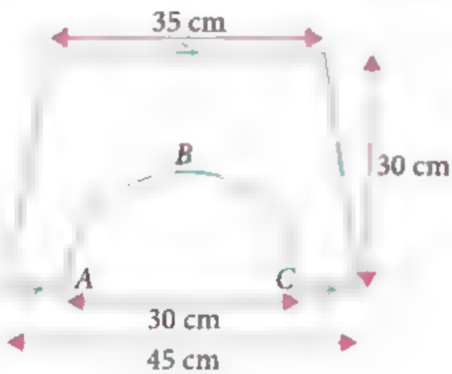


Fig. (b)

Calculate the percentage increase or decrease in the area of the cross section.

Exercise 12C

9. Raju cut out the various shapes (shown in Fig. **(a)**) from an A4-sized coloured paper to make a 2D representation of the Eiffel Tower (shown in Fig. **(b)**) when pieced together. *AB*, *CD*, *EF*, *GH*, *IJ*, *KL* and *NP* are parallel to one another. Arc *LMN* is a semicircle. If an A4-sized paper has dimensions 210 mm by 297 mm, find the percentage of paper he used to make this representation.

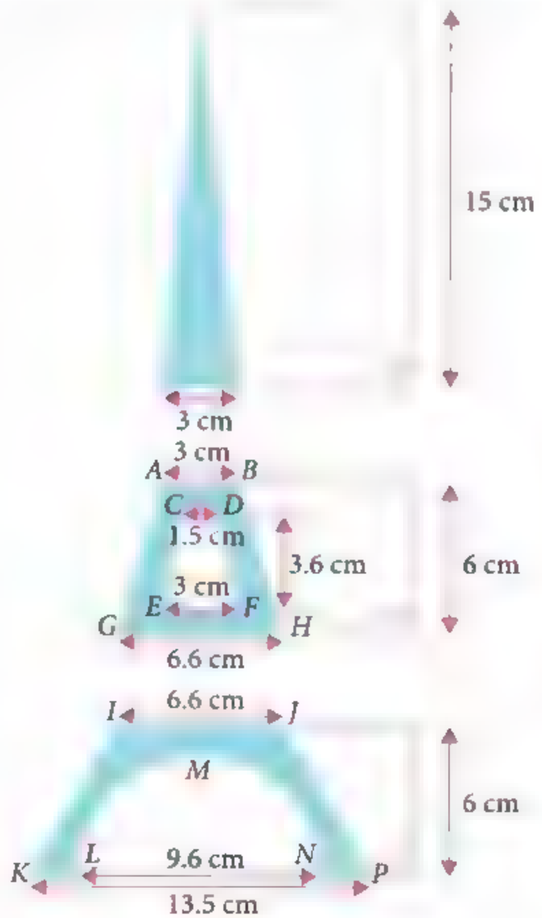


Fig. (a)

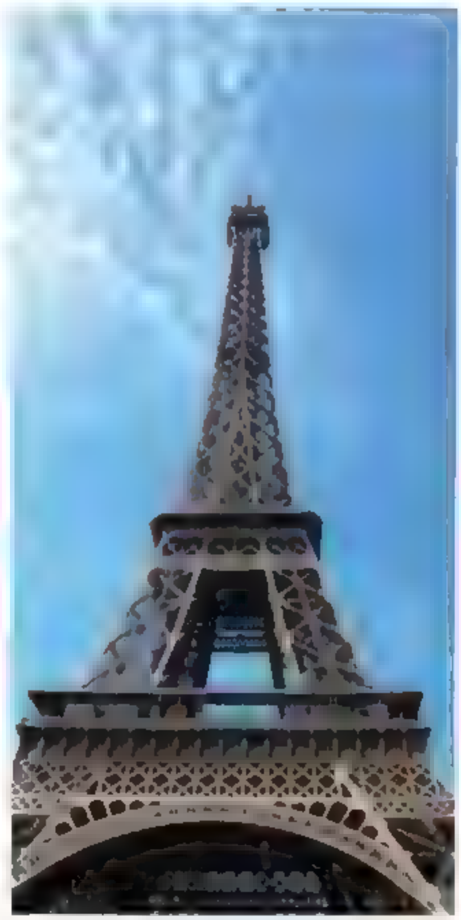
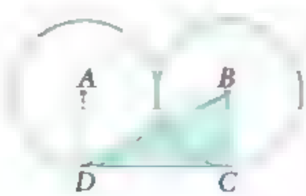


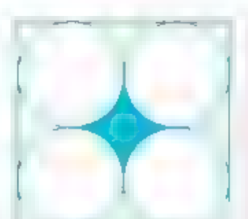
Fig. (b)

10. In the figure, two identical circles, with centres at *A* and *B* respectively, have an area of 0.785 cm² each. If *A* is directly above *D*, find the area of the shaded region.

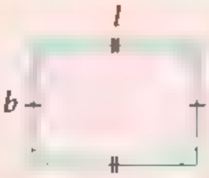
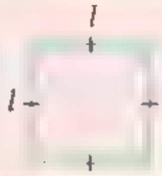
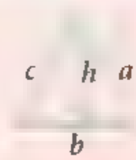


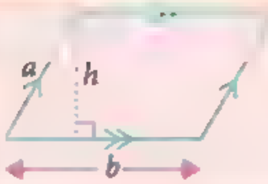
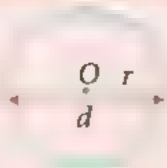
11. A goat, tethered by a rope 1.5 m long, is able to eat a square metre of grass in 14 minutes. Find the time it needs to eat all the grass within its reach.

12. The figure shows four identical circles inside a square of side 28 cm. Find the perimeter and area of the shaded region.



In this chapter, we see how we can make use of the areas and perimeters of plane figures that we know to determine the areas and perimeters of other figures. This enables us to extend our understanding of plane figures and work with new combinations of plane figures. Geometrical diagrams of plane figures are used to **model** real-world objects. For example, 2D animations such as cartoon characters (e.g. Mickey Mouse), or even logos (e.g. Apple's logo) are designed based on basic shapes. The technique of decomposing and composing figures to form new figures is useful to solve real-world problems involving mensuration, such as the design of objects or buildings. Understanding the basic plane figures and their properties help in the construction of diagrams, which approximate the actual object during the design process. This is a cost-effective way of rapid prototyping, which saves money and time as compared to building actual objects to test our designs. All these applications rely on a clear understanding of different **measures** of area and perimeter, especially the conversion between different units of measurement. Fluency in expressing area and perimeter using appropriate units is critical to reduce errors during prototyping and in the actual construction of objects and buildings.

Name	Rectangle	Square	Triangle
Figure			
Perimeter	$2(l + b)$	$4l$	$a + b + c$
Area	lb	l^2	$\frac{1}{2}bh$

Name	Parallelogram	Trapezium	Circle
Figure			
Perimeter	$2(a + b)$	$a + b + c + d$	$2\pi r = \pi d$
Area	bh	$\frac{1}{2}(a+b)h$	πr^2

- Think of a real-world object that you can use the above formulae to find the perimeter and area of.

CHAPTER 13

Statistical Data Handling

How tall are the students in your school? This is not an easy question to answer for several reasons. Firstly, the height of each student in the school is different. Secondly, it would be tedious to list down the height of every student in the school. Thirdly, how can we process the various heights to come up with a single value?

Questions like these are statistical questions. Answering them would require the collection, analysis, interpretation, and representation of data. Many questions in the real world are statistical questions. The following are some examples.

- How many students borrow books from the library each month?
- How did students do for the end-of-year exams?
- What is the average income of Pakistanis?

In this chapter, we are going to embark on another exciting branch in mathematics statistics, which is the science of collecting, organising, analysing and interpreting data.

Learning Outcomes

What will we learn in this chapter?

- What frequency tables, pictograms, bar graphs, and pie charts are
- How to collect, classify, tabulate, display, analyse and interpret data so as to make inferences, predictions and informed decisions
- Why different statistical diagrams are appropriate for different purposes
- Why some statistical data or diagrams can lead to misinterpretation

Introductory Problem

Fig. 13.1 shows a bar chart of the sales of single bedroom apartments in the first three months of a particular year.

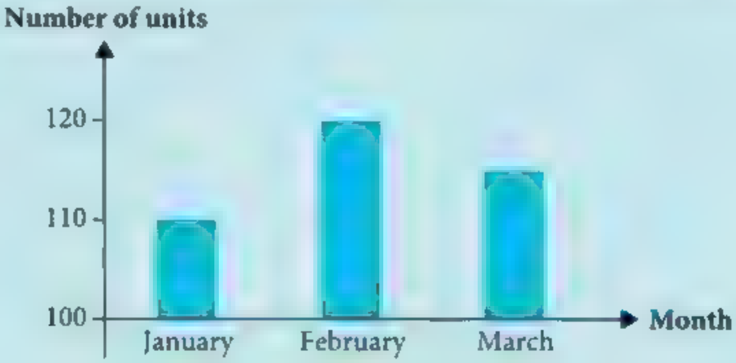


Fig. 13.1

After a quick glance at the bar chart, Bernard says, "The number of units sold in February is twice the number of units sold in January." Is Bernard correct? Explain your answer.

In this chapter, we will learn how to interpret statistical diagrams such as pictograms (or picture graphs), bar charts (or bar graphs), and pie charts (or pie graphs). We will also discuss the advantages and disadvantages of each of these diagrams, and how data and statistical diagrams can be misinterpreted. Finally, we will learn how to conduct a statistical investigation in order to make informed decisions.

13.1 Frequency table

A new school canteen vendor surveys 500 students in the school to find out which fruit students like the most. In the survey, each student can only choose one fruit. The results of the survey are collated in Table 13.1.

Statistical data refer to the information collected. In this case, the data of 500 favourite fruits of 500 students can be divided into 5 *categories*: apple, pear, honeydew, watermelon and orange.

Fruit	Apple	Pear	Honeydew	Watermelon	Orange	Total
Number of students	100	50	75	150	125	500

Table 13.1

Frequency in each category: *number of students* who have chosen the category, e.g. frequency for 'Apple' is 100

Total frequency: total number of students across all categories

Table 13.1 is called a **frequency table**. It is used to *organise* and *display* the data. Therefore, a frequency table can also be considered a **statistical diagram**. However, it is *not a statistical graph* like a pictogram or a bar chart.

Attention

The word 'data' is the plural form. Its singular form is 'datum', which is seldom used. Instead, we use the term '*data point*' or '*data value*' (if the data are numerical values) to refer to a datum. The whole set of data is called the '*data set*'.

Statistical diagrams, such as a frequency table, organise and display the data in a manner that summarises and communicates the important characteristics of the data, e.g. which fruit was most frequently chosen.

Frequency table	
Advantages	**Disadvantages**
Used primarily to organise data collected	Display of data not as visual and appealing as statistical graphs
Easy to read the exact frequency for each category	Harder to compare frequencies across different categories than some statistical graphs

Attention

Since a frequency table is not a statistical graph, we use the term '**statistical diagrams**' to include both the frequency table and *statistical graphs* such as pictograms and bar graphs.

Statistical data can be **categorical** (such as the types of fruits) or **numerical**. Numerical data can be **discrete** or **continuous**.

Types of statistical data	
Categorical data	**Numerical data**
• Represents characteristics which can be grouped, e.g. gender, types of fruits • Relative size of categorical data cannot be compared	• Consists of numbers, e.g. monthly income, height of students • Relative size of numerical data can be compared, e.g. 180.5 cm is taller than 180.1 cm.
• Most data cannot be ordered, except for data such as 'agree', 'neutral' and 'disagree'	• All data can be ordered
• Displayed using pictograms, bar graphs and pie charts	• Displayed using line graphs and other types of statistical diagrams (which we will learn in Book 3)

Note that some categorical data, such as months, years or bus numbers, can appear to be numerical data. These 'numerical data' are treated as '*labels*'.

In Sections 13.2 to 13.4, we will learn the use of statistical graphs to present categorical data.

Purposes and appropriateness of pictogram

Part 1

The data in Table 13.1 is displayed as a **pictogram (or picture graph)** in Fig. 13.2. This is a *pictogram with scales* because the *key* (or legend) indicates that each picture in the pictogram represents more than one student.

Students' favourite fruit

Key:
Each picture represents 20 students.

Fig. 13.2

A pictogram is another statistical diagram used to display statistical data. The key (or legend) indicates the number of items represented by each picture and the number of pictures represents the frequency of each category. The size of each picture should be the same and the pictures should be equally spaced out.

Proportionality
If each picture of a pictogram represents 20 students, half a picture will be used to represent 10 students and one-quarter of a picture will be used to represent 5 students. We learnt in Chapter 8 that this is called proportionality: the size of a picture is directly proportional to the number of students it represents (or the frequency).

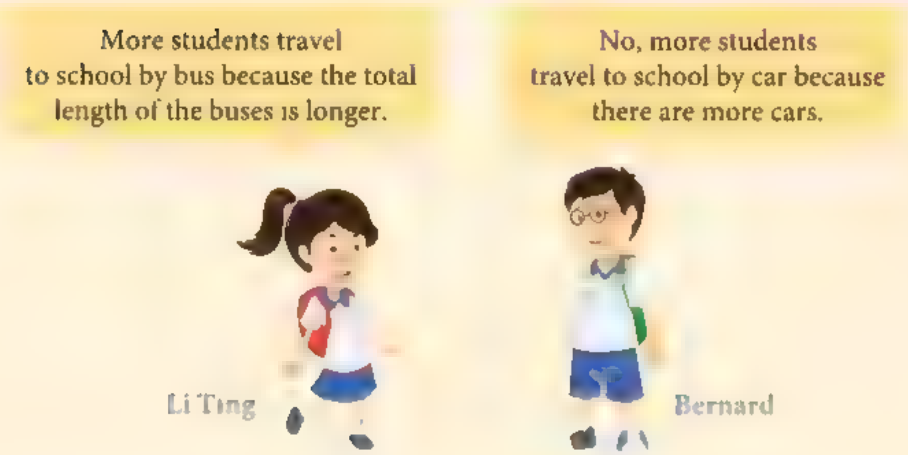
1. Which is the favourite fruit among the 500 students? How is this different from the favourite fruit of each student?
2. How many students chose pear? Is it easy to read from the pictogram?
3. How many students chose honeydew? Is it easy to read from the pictogram?
4. What is one problem of using a pictogram to display the above data as suggested by your answer to Question 3?

Part 2

The pictogram in Fig. 13.3 shows the number of students in a class who go to school either by bus or by car.

Key:
Each picture represents 5 vehicles.

Fig. 13.3



5. Is Li Ting or Bernard correct? Explain your answer.
6. How do you modify the pictogram to avoid a misinterpretation of data?
7. Does it look odd if the length of a bus is the same as the length of a car in a pictogram?
8. What are some other advantages or disadvantages of a pictogram?

From the above Class Discussion, we observe the following:

Pictogram

Advantages	Disadvantages
Colourful and appealing	Difficult to draw the pictures, unless one uses a software
	Difficult to draw a fraction of a picture, which may also result in misinterpretation
	Actual frequency in each category may be distorted and misinterpreted due to the different sizes of the pictures in different categories

The pictogram shows the profits earned by a company in each year from 2017 to 2022.

Profits earned by a company

Key:

represents $1 000 000

(i) What was the profit earned by the company in
 (a) 2020, (b) 2022?

(ii) In which year did the company earn the least profit? How much did the profit decrease that year as compared to the previous year?

13.3 Bar graph

Let us look at the pictogram in Fig. 13.2 again. If we draw a bar over each row of pictures as shown in Fig. 13.4, we will obtain a horizontal bar graph (or bar chart) as shown in Fig. 13.5 on page 339. We can also display the bar graph vertically as shown in Fig. 13.6.

Student's favourite fruit

Key:
Each picture represents 20 students.

Fig. 13.4

Students' favourite fruit

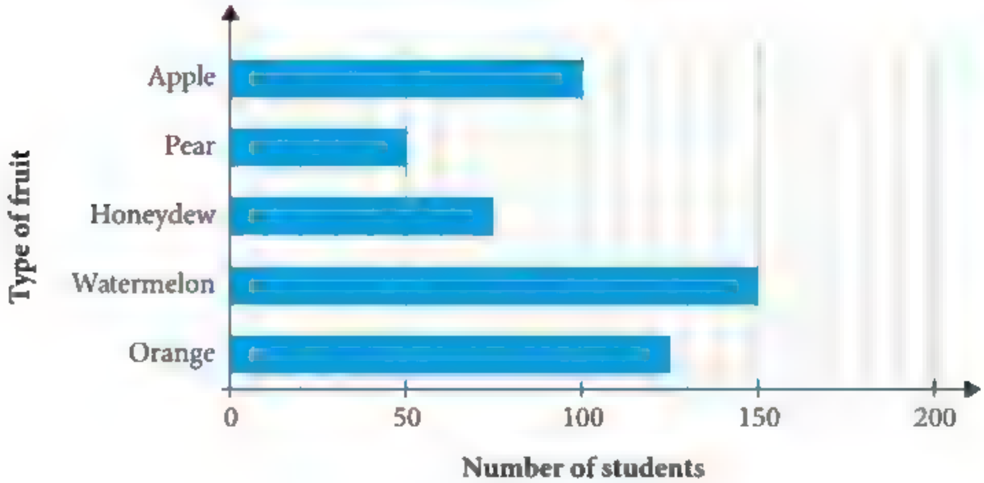


Fig. 13.5

A bar graph is another statistical diagram used to display statistical data. The frequency of each category is represented by the length of the bar. The bars must be of the same width. A space is left between two consecutive bars to distinguish between the categories.

Students' favourite fruit

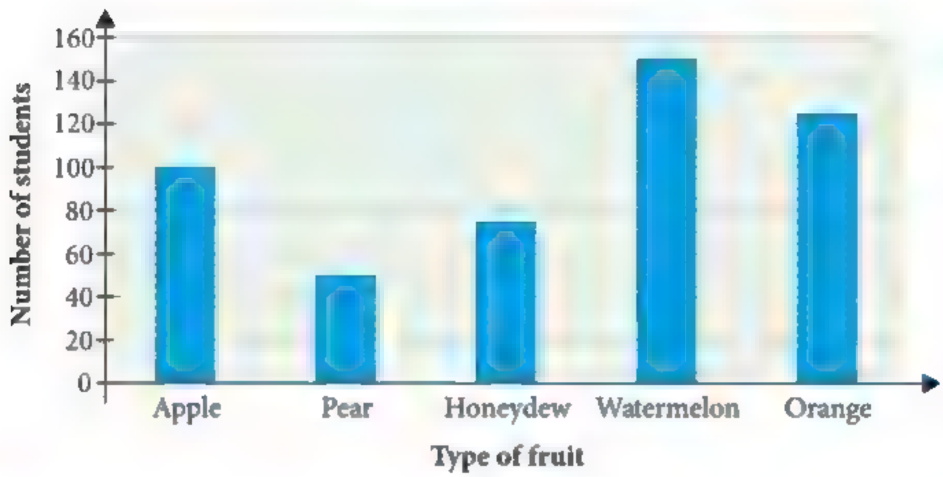


Fig. 13.6

Proportionality

The length of a bar in a bar graph is directly proportional to the frequency, provided the frequency axis starts from 0. For example in Fig. 13.5 and 13.6, the length of the bar for 'Apple' which represents 100 students is twice as long as that for 'Pear' which represents 50 students.

Purposes and appropriateness of bar graph

Part 1: Bar graph vs. pictogram

Compare the pictogram in Fig. 13.2 on page 336 and the bar graph in Fig. 13.6.

1. What advantages does a pictogram have over a bar graph?
2. What advantages does a bar graph have over a pictogram?

Part 2: Introductory Problem Revisited

The *frequency axis* is the vertical axis of a vertical bar graph, or the horizontal axis of a horizontal bar graph.

3. What did you learn about the frequency axis of a bar graph from the Introductory Problem?
4. Do you think we should always draw the frequency axis of a bar graph to start from zero? Why?

Attention

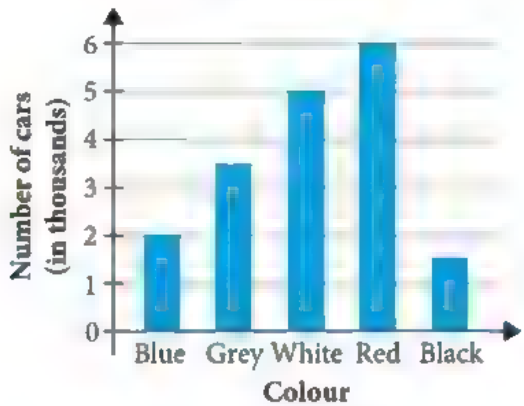
For an in-depth discussion of Question 4, see Section 13.5 Class Discussion Question 2.

From the Class Discussion on page 339, we learn that the **frequency axis** of a bar graph should start from zero to avoid distortion and misinterpretation of data.

Bar graph	
Advantages	**Disadvantages**
Easier to draw than a pictogram	Less colourful and appealing than a pictogram
Do not have to draw a fraction of a picture	More abstract because frequency is represented by length of each bar
Not distorted by different sizes of pictures	If the frequency axis does not start from zero, the data may be distorted or misinterpreted

1. The bar graph shows the number of cars of different colours sold in one year in a city.

Sales of cars

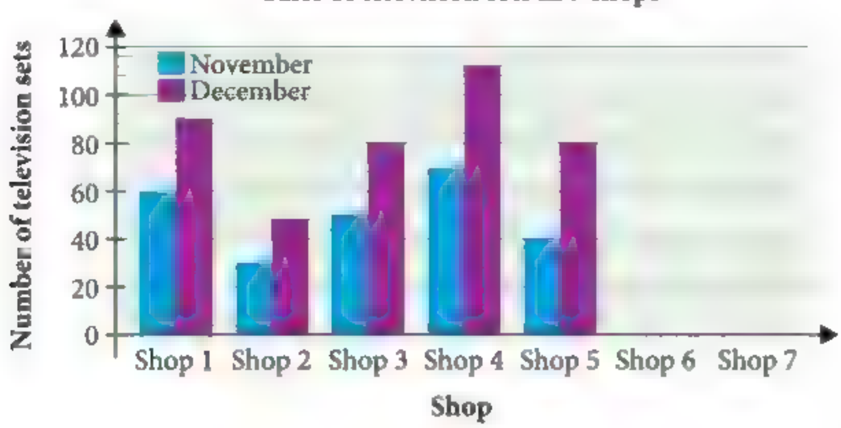


(i) Which is the least popular colour?

(ii) Shaha says that there is a mistake in the bar graph because 3.5 grey cars and 1.5 black cars do not exist in the real world. Do you agree with him? Explain your answer.

2. A company owns seven electrical shops. Study the bar graph.

Sales of television sets in 7 shops



(i) Complete the bar graph using the data given in the table.

	Shop 6	Shop 7
November	64	70
December	88	96

(ii) Find the total number of television sets sold in the seven shops in
 (a) November, (b) December.

(iii) Express the total number of television sets sold in the seven shops in November as a percentage of the total number of television sets sold in the seven shops in November and December.

(iv) (a) Express the total number of television sets sold in Shop 7 in November and December as a percentage of the total number of television sets sold in the seven shops in November and December.
 (b) The company would like to close down one shop due to insufficient cash flow. Based on the number of television sets sold in November and December, the manager proposed to close down Shop 7. Do you agree with the manager? Explain your answer.

(v) In which month did the company perform better in terms of sales? Explain your answer.

[illegible] | Intermediate | Basic

Exercise [illegible]

1. The pictogram illustrates the number of registered buses in a country from May to September 2022.

Number of registered buses

Key:
 represents 50 buses

(i) In which two months were the greatest number of buses registered? Estimate the number of buses registered in each of those months.

(ii) Estimate the total number of buses registered from May to September 2022.

(iii) If the registration fee for each bus in 2022 was \$220, estimate the total amount collected from the registration of buses from May to September 2022.

(iv) Estimate the percentage decrease in the number of buses registered from July to August 2022.

Exercise 13A

2. The table shows the number of students who play volleyball, basketball and tennis respectively.

Sport	Volleyball	Basketball	Tennis
Number of students	40	60	50

(i) Complete the pictogram.

Key:
represents 10 students

(ii) Find the ratio of the number of students who play volleyball to the number of students who play tennis.

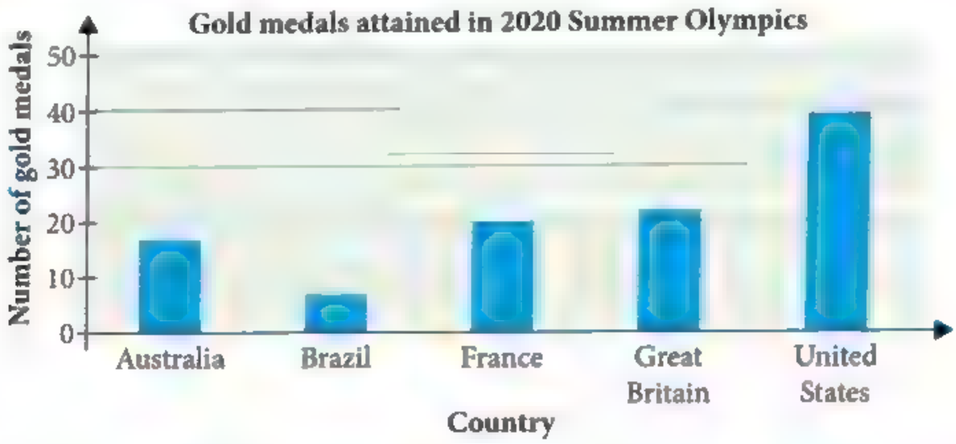
(iii) Express the number of students who play tennis as a percentage of that who play basketball.

3. The table shows the number of copies of a newspaper distributed to households in each year from 2017 to 2021.

Year	2017	2018	2019	2020	2021
Number of copies (in thousands)	250	275	290	315	280

Use a bar graph to illustrate the above information.

4. The bar graph shows the number of gold medals attained by five countries during the 2020 Summer Olympics.

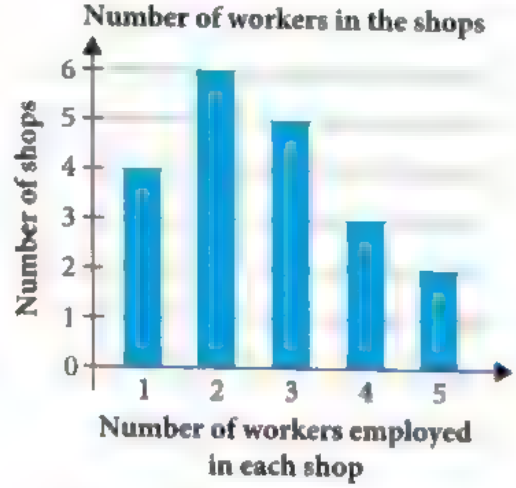


(i) What was the total number of gold medals attained by the five countries?

(ii) What percentage of the gold medals amongst these five countries were attained by France?

Exercise

5. The bar graph illustrates the results of a survey conducted on shops in a town.

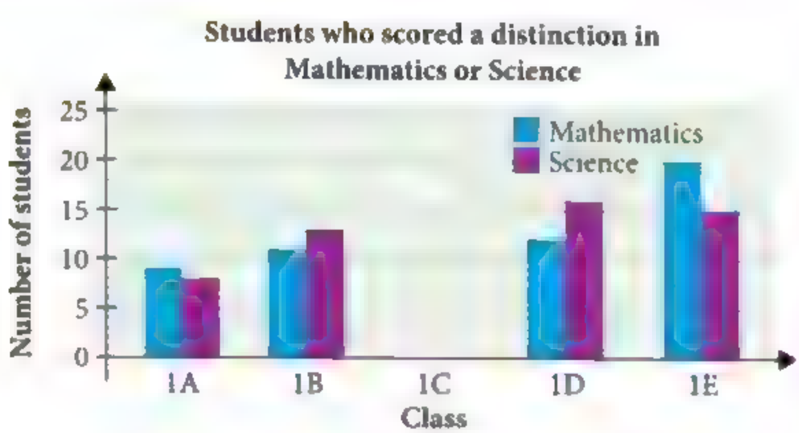


(i) Find the total number of workers employed in the town.

(ii) Express the number of shops hiring 3 or more workers as a percentage of the total number of shops in the town.

6. Study the table and bar graph.

Class	Class 1A	Class 1B	Class 1C	Class 1D	Class 1E
Number of students who scored a distinction in Mathematics	9	11	16	12	20
Number of students who scored a distinction in Science	8		12		15



(i) Complete the table and the bar graph.

(ii) Find the total number of students in the 5 classes who scored a distinction in

(a) Mathematics, (b) Science.

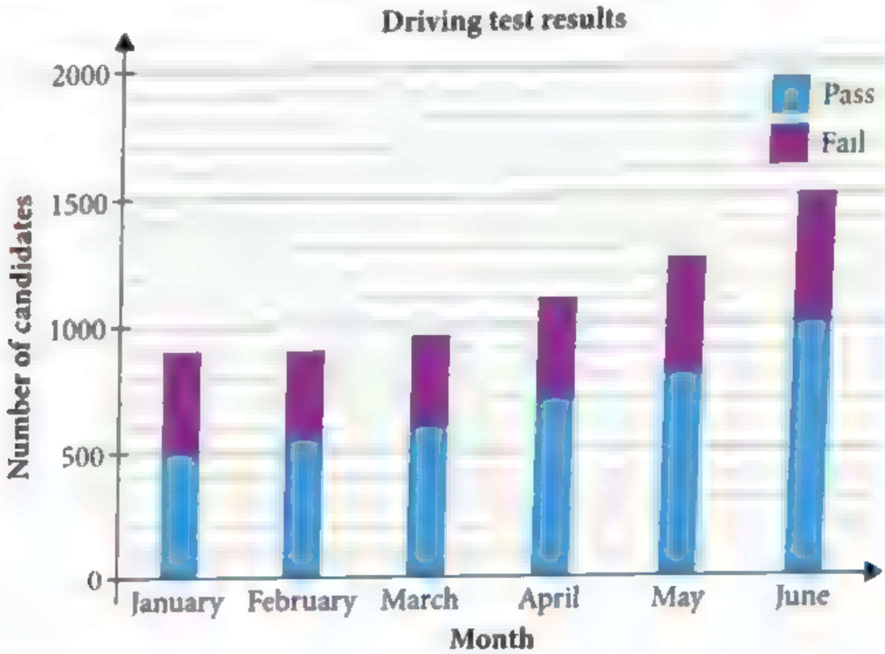
(iii) Express the number of students who scored a distinction in Mathematics in Class 1D as a percentage of the total number of students who scored a distinction in Mathematics in the 5 classes.

Exercise 13A

(iv) If there are 40 students in Class 1D, find the percentage of students in the class who scored a distinction in Science.

(v) Is Waseem correct to say that there are 35 students in Class 1E? Explain your answer.

The graph shows the number of candidates who took a practical driving test in each month.



(i) How many candidates took the test in March?

(ii) How many candidates failed the test in June?

(iii) Express the number of candidates who failed the test in June as a percentage of the total number of candidates who failed the test in the six months.

(iv) Comment on the trend in the percentage of successful candidates over the six months. Provide a reason for this trend.

13.4 Pie chart

Fig 13.7 shows the information in Table 13.1 represented in a pie chart (or pie graph).

Fig. 13.7

> A pie chart is another statistical diagram used to display statistical data. The circle is divided into sectors where the area (or the angle) of the sector represents the frequency of each category. The frequency or angle for each category may be labelled in the corresponding sector.

Each sector of the pie chart represents a category in Table 13.1. How do we construct these sectors to present the information in Table 13.1 accurately?

Construction and usefulness of pie chart

Part 1: Construction

We will use the same set of data in Section 13.1 on the favourite fruits of 500 students.

To draw a pie chart, we have to calculate the angle of each sector, which represents the number of students in that category. The full circle (or 360°) represents 500 students (total frequency) and each sector angle is directly proportional to the frequency of that category.

500 students are represented by 360°.

1 student is represented by $\frac{360^\circ}{500}$.

100 students are represented by $\frac{360^\circ}{500} \times 100$

$= \frac{100}{500} \times 360^\circ$

Therefore, sector angle $= \frac{\text{frequency of category}}{\text{total frequency}} \times 360^\circ$.

> **Proportionality**
>
> The sector area (or sector angle) of a pie chart is directly proportional to the frequency. For example, in the survey of 500 students, the full circle (or 360°) represents 500 students. Half a circle (or 180°) will represent 250 students and a quarter of a circle (or 90°) will represent 125 students. In Fig. 13.8**(b)**, the sector for 'Apple' is $\frac{100}{500}\left(=\frac{1}{5}\right)$ of the whole circle and the sector angle for 'Apple' is $\frac{1}{5}$ of 360°, which is 72°.

1. Copy and complete Table 13.2.

Fruit	Apple	Pear	Honeydew	Watermelon	Orange
Number of students (Frequency)	100	50	75	150	125
Angle of sector	$\frac{100}{500} \times 360^\circ = 72^\circ$				

Table 13.2

The circle in Fig. 13.8(a) represents the total number of students surveyed, i.e. 500 students.

A protractor is used to construct a sector with an angle of 72° inside the circle as shown in Fig. 13.8(b).

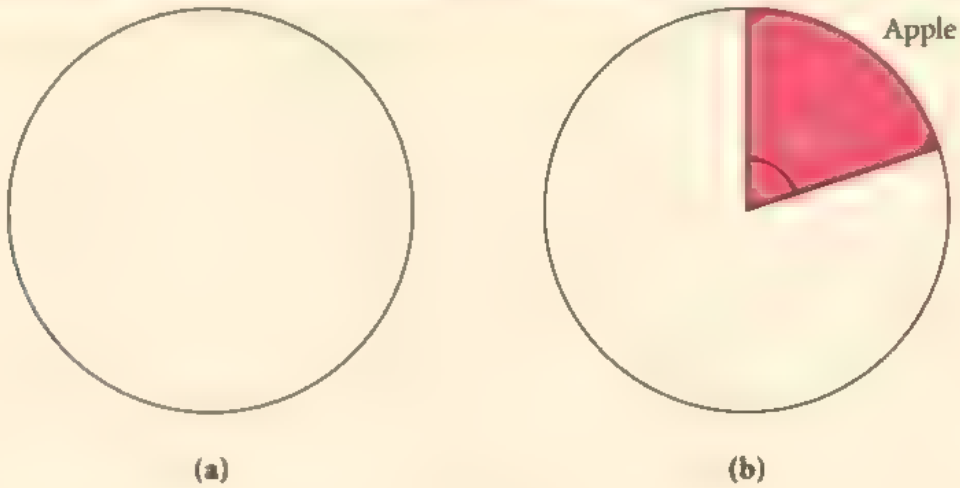


Fig. 13.8

2. Construct all the other sectors of the pie chart.
3. Since the sector area (or sector angle) represents the number of students in that category, the larger the sector size (or sector angle), the larger the number of students in that category.
   Look at the pie chart. Which fruit is the most popular among the 500 students? Why?
4. (i) Look at the pie chart. What fraction of the 500 students like orange the most?
   (ii) Are you able to obtain the fraction in part (i) easily by looking at the corresponding bar graphs in Fig. 13.5 or Fig. 13.6 on page 339?

Part 2: Advantages and disadvantages

5. What advantages does a pie chart have over a pictogram or a bar chart?
6. What disadvantages does a pie chart have?

Fig. 13.9 shows the monthly expenditure of Raju presented in a 3-dimensional pie chart and a 2-dimensional pie chart.

Fig. 13.9

7. Based on the 3-dimensional pie chart, which item does Raju appear to spend the most on?
8. Based on the 2-dimensional pie chart, which item does Raju actually spend the most on?
9. Suggest a reason to explain the discrepancy between your answers in Questions 7 and 8.
10. When interpreting a 3-dimensional pie chart, what should you take note of?
11. Is it easy to tell from the 2 dimensional pie chart whether Raju spends more on food or on utilities?

From the Class Discussion on pages 345 and 346, we observe the following:

Pie chart	
Advantages	**Disadvantages**
Easier to compare relative size of each category with the whole, e.g. one quarter, or more (or less) than half	More abstract than pictogram because frequency is represented by sector area (or angle)
	Harder to construct than a bar chart
	May look cluttered if there are many categories (or sectors)
	Difficult to compare two sectors of about the same size
	Sizes of sectors may be distorted and misinterpreted in a three-dimensional pie chart
	Difficult to determine the frequency of each category, unless it is stated inside each sector

The table shows Imran's expenditure on a holiday.

Item	Food	Shopping	Hotel	Air ticket	Others
Amount spent	\$1000	\$1200	\$400	\$1200	\$200

Construct a pie chart using the data from the table.

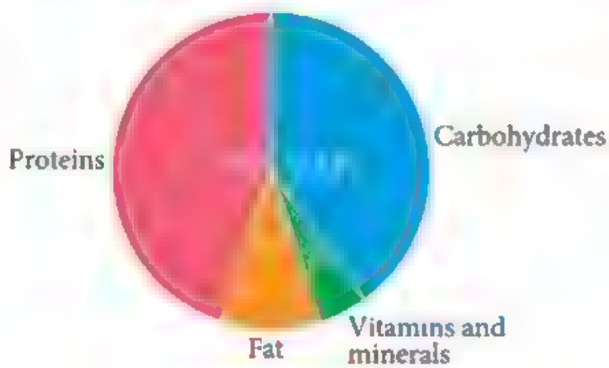
Problem involving pie chart

The pie chart shows the nutritional composition of a food product.

(i) Calculate the value of x.

(ii) Express the amount of fat as a percentage of all the components given.

(iii) Given that a food product contains 120 g of carbohydrates, calculate the mass of the food product.



(i) $144° + 9x° + 2x° + x° = 360°$ ($\angle$s at a point)

$$12x = 360 - 144$$
$$= 216$$
$$x = 18$$

(ii) Angle of sector representing fat = $2 \times 18°$
$= 36°$

Required percentage $= \frac{36°}{360°} \times 100\%$
$= 10\%$

(iii) Angle of sector representing carbohydrates = 144°

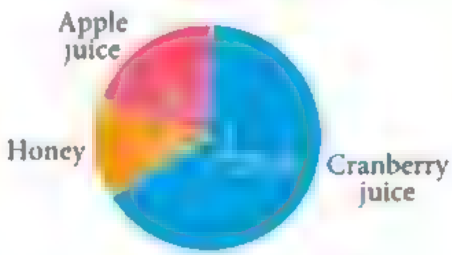
144° represent 120 g

1° represents $\frac{120}{144}$ g

360° represent $\frac{120}{144} \times 360 = 300$ g

$\therefore$ mass of food product = 300 g

The pie chart shows the composition of a jar of fruit punch.



(i) Find the value of x.

(ii) Express the amount of apple juice in the fruit punch as a percentage of all the components in the fruit punch.

(iii) Given that a jar of fruit punch contains 759 ml of cranberry juice, find the amount of fruit punch in the jar.

1. After learning the advantages and disadvantages of the use of different kinds of statistical diagrams to represent a set of data, how do I determine the most suitable statistical diagram to display a set of data in a real-world context?
2. What are some aspects of a statistical diagram to pay attention to so that I will not be misled by it?

Choose the most suitable statistical graph(s) to display the data for each of the following real-world scenarios. Explain the reason(s) for your choice.

(a) The distribution of birthday months in your class.

(b) The number of Secondary 1 students who travel to school by the various modes of transport, e.g. by bus, by car, by metro or by walking.

(c) The percentages of Secondary 1 students who prefer the following different types of drinks: iced lemon tea, orange juice, chocolate milk, isotonic drink and plain water.

13.5 Evaluation of statistical representations

As we have learnt in the previous sections, certain misinterpretations can arise from inappropriate uses of statistical diagrams. In this section, we will examine some other issues that can arise from the collection, organisation, display and interpretation of data.

Evaluation of statistical representations

Part 1: Collection of data

Read the article and answer the questions.

NEWS

Zidane Named Best European Footballer in Last 50 Years

PARIS: In a UEFA (Union of European Football Associations) website poll in 2004, Zinedine Zidane was named Europe's best footballer in the past 50 years. He obtained 123 582 votes, followed by Franz Beckenbauer with 122 569 votes and Johan Cruyff with 119 332 votes.

1. Do you know who the three footballers are? Your teacher will take a poll in your class to find out the number of students who know each of the three footballers.
2. How did UEFA conduct the poll? Were the voters who took part in the poll representative of all football fans? Explain your answer.
3. If older football fans were to participate in the poll, do you think Zidane would have come in first place? Justify your opinion.

   Zidane was famous in the 1990s while Beckenbauer and Cruyff were at the peak of their careers in the 1970s.
4. What lesson can you learn about the choice of a sample during data collection?

Part 2: Organisation of data

Read the article and answer the questions.

NEWS

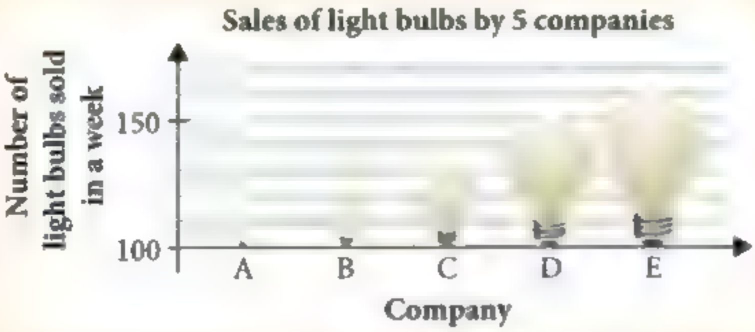
Most Number of Complaints Received Against Banks and Insurance Firms for the First Time

QASVILLE: The Customers Organisation of Qasville revealed that they have received the most number of complaints against banks and insurance firms for the first time, i.e. 1416 complaints in 2012. Coming in second was timeshare companies with 1238 complaints, followed by motor vehicle companies with 975 complaints.

5. Which three types of companies received the most number of complaints?
6. A statistician commented, "It is possible that timeshare companies receive the most number of complaints." Discuss with your classmates why this comment may be true.
   Hint: How were the data (number of complaints) organised?
7. What lesson can you learn about the organisation of statistical data?

Part 3: Display of data

The graph in Fig. 13.10 shows the number of light bulbs sold by 5 companies in a week. Study the graph and answer the questions below.



8. Company E claims that it has sold twice as many light bulbs as Company C because the light bulb corresponding to Company E in the graph is twice as big as that of Company C. Is the claim valid? Explain your answer.
9. What lesson can you learn about the display of statistical data?

Part 4: Interpretation of data

Read the article and answer the questions.

NEWS

Employees' Satisfaction

QASVILLE: In a survey conducted among 300 employees of a company, only 40% of them were not satisfied with working in the company. Therefore, the survey concluded that the employees were satisfied with the company and that the company was a good place to work in.

10. How did the survey arrive at the conclusion as stated in the article?

11. Although only 40% of the employees were not satisfied with working in the company, can you conclude that the employees were satisfied with the company and that the company was a good place to work in?
12. In order to pass a Constitutional Amendment Bill under Pakistan's Federal Parliamentary System, at least two-thirds of the members of each House must agree on it. Why is it that a simple majority is not enough in this case? Explain your answer.
13. What lessons can you learn about the interpretation of statistical data?

Part 5: Ethical issues

From Parts 1 to 4, we have seen some examples of statistical misuse. Poor use of statistics can be found everywhere, e.g. in magazines and advertisements. Have you ever encountered such instances? Discuss with your classmates why people inadvertently or intentionally use statistics to mislead others. Why is this unethical?

Exercise 13B

Cheryl says, "Statistical data do not lie. It is the interpretation of the data that can lie."
Do you agree? Explain.

13.6 Statistical investigation

In this section, we will consolidate what we have learnt in the previous sections by illustrating how a statistical investigation can be carried out in real life. Fig. 13.11 shows a model of the four stages of a **statistical investigation**

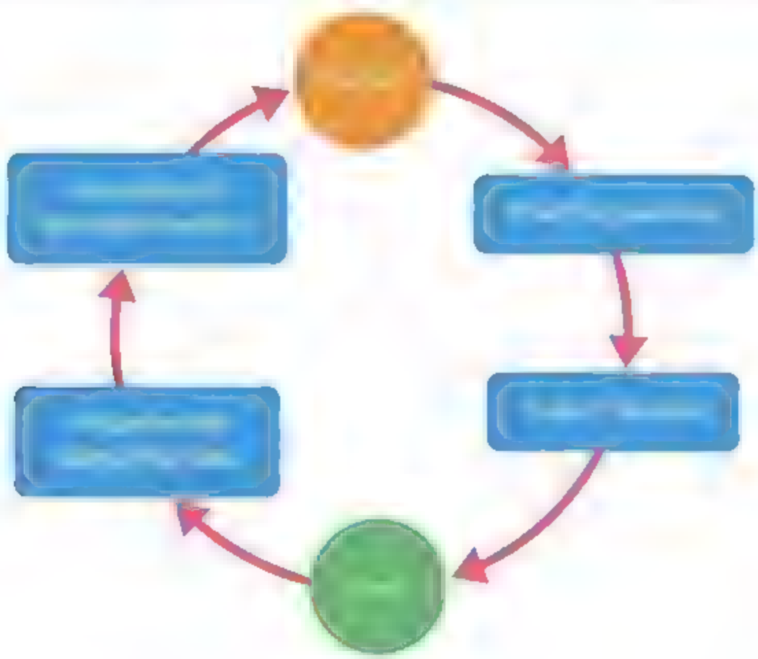


Fig. 13.11

Problem

First, there must be some kind of a problem that warrants a statistical investigation. For example, the student council of a school wants to organise a Games Day for students to play some sports.

1. What are some decisions that the student council has to make?

2. Suppose the student council decides that they have the resources to organise only three sports. In order to attract students to take part, the three sports should be the three most popular sports among the student population. What are some ways to obtain this information?

> **Attention**
> When writing a report for a statistical investigation, we should provide some background information of the problem and the question for the investigation.

Some methods of finding the answer to Question 2 may not involve a statistical investigation. Suppose the student council decides to conduct a statistical investigation. Then they can pose a question for the investigation.

A. Pose the question

The question is, "What are the three most popular sports among the students in this school?"

B. Collect the data

Method

To answer the question, the student council has to first decide on the **method** of data collection. Suppose the student council decides to *survey* the students in the school.

3. Is it practical to survey all the students in the school? If not, what else can you do?

> **Attention**
> There are different methods of data collection, e.g.
> - Survey
> - Interview
> - Experiment, e.g. an experiment can be carried out to find out the average lifespan of light bulbs
> - Observation, e.g. scientists use observations to study behavioural patterns of different species

Sample

Since it is too time consuming to survey the entire student population in the school, the student council has to choose a **sample** from the student **population** to survey.

4. The student council may choose a *convenient sample*, e.g. survey students in the school canteen during recess. But this sample may not be a *representative sample* of the student population. Suggest a reason why surveying students in the school canteen during recess may not include students who may be more likely to take part in the Games Day.

5. In this case, do you want the sample to be representative of the entire student population in the school, or representative of students who may be more likely to take part in the Games Day?

6. How would you decide on the sample so that it has a higher chance of being representative?

> **Attention**
> An important criterion to decide on who to sample is, "*Is* [illegible]

The student council also has to decide on the *sample size*, i.e. the number of students to survey.

7. If the sample size is too small, what will it affect?

8. If the sample size is too large, what will it affect?

Test instrument

The student council has to design a **test instrument** to collect the data. For a survey, the test instrument can just be a *questionnaire*. Fig. 13.12 shows an example of a simple questionnaire.

From the list below, put a tick next to the sport which you like the most.
You should only select one sport.

☐ Soccer ☐ Volleyball ☐ Basketball

☐ Hockey ☐ Netball

Fig. 13.12

> **Attention**
> The survey question can be slightly different from the question of the statistical investigation posed in

9. How do you decide on a list of sports in the questionnaire for the survey participants to choose from?
10. How do you ensure that the list does not miss out any popular sports?
11. Do you want the participants to choose three sports that they like the most, or just one sport? What are the pros and cons of either decision?

Data

The results of the questionnaire form the data. To analyse and interpret the data, the student council has to first organise the data. They may also choose to display the data using a statistical graph.

C. Organise and display the data

Organise the data

To organise the data, the student council has to classify the data into **categories**. In this case, the categories will be according to the types of sports. A **frequency table** with a 'Tally' column as shown in Table 13.3 can be set up. Each tally mark against a sport corresponds to one student who chose that sport.

Sport	Tally	Number of students
Soccer	[illegible]	176
Volleyball	[illegible]	144
Basketball	[illegible]	
Hockey	[illegible]	
Netball	[illegible]	
Total		

Table 13.3

12. Why do you think the tally marks are organised in groups of fives ( ) with the fifth tally mark crossing the first four tally marks?
13. Copy and complete Table 13.3.
14. What is one way to check if you have recorded or counted the tally marks wrongly?

Display the data

The frequency table is not only used to organise the data, but also as a statistical diagram to display the data. For example, we can determine the three most popular sports from the number of students in the last column of Table 13.3. However, it is harder to see the distribution of the data and it is less visual, in comparison to using a statistical graph. Suppose the student council decides to use a spreadsheet on a computer to create a bar graph to display the data as shown in Fig. 13.13.

Different statistical diagrams are used to display different types of data and they focus on different characteristics of the data these are important considerations for the choice of an appropriate diagram to display the data.

Internet Resources

Search the Internet for instructions on how to create statistical diagrams using a spreadsheet.

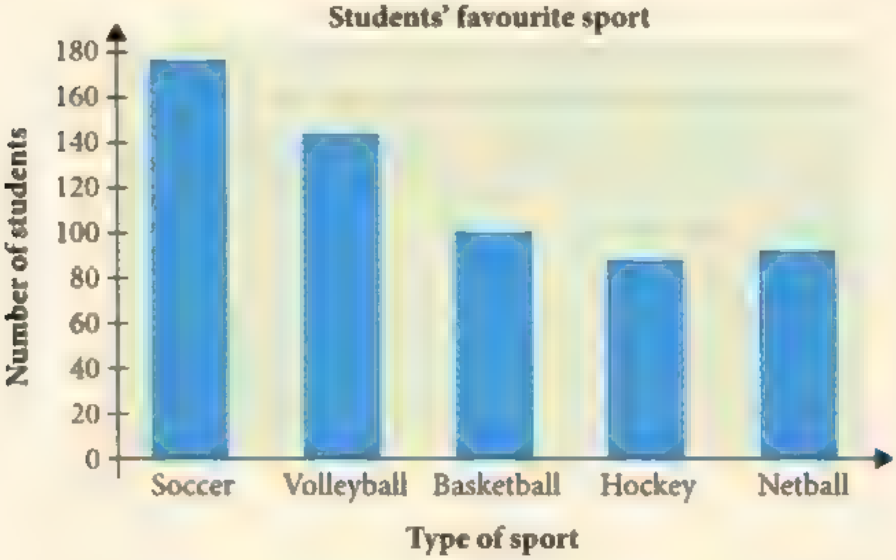


Fig. 13.13

15. Do you think a bar graph is an appropriate choice of a statistical graph to display the data? Should you choose a pictogram, a pie chart or a line graph instead? Explain.

D. Analyse and interpret the data

Analyse the data

To analyse the data, the student council can use the frequency table or the bar graph.

16. Do you think it is easier to determine the top three sports from the frequency table or the bar graph?
17. Name the top three sports as indicated by the survey.

Interpret the data

It is obvious that the top two most favourite sports are soccer and volleyball. However, the difference between the numbers of students who chose basketball and those who chose netball is small.

18. Is it possible for netball to be more popular than basketball in the entire student population? Explain.
19. Should the student council choose basketball or how can they verify what the third most popular sport is?

Attention

We have to be careful how we interpret the data, especially when the frequency for two or more categories is quite similar.

Answer the question posed

The answer to the question, "What are the three most popular sports among the students in this school?" is: soccer, followed by volleyball and then basketball.

The answer to the question may include **assumptions**. In this case, we *assume* that the sample is representative and that the student council does not want to further examine if netball is more popular than basketball. What other assumptions can you think of?

Communication of results

It is important to be able to conduct a statistical investigation, answer the question posed, and to communicate the results of the investigation to other relevant parties. For example, the senior management of the school may want to know how the student council decided on the three sports. We should learn how to write a statistical report or prepare a presentation that includes all the key features in this Class Discussion.

Discuss with your classmates some possible issues that your school is facing, e.g. the lack of variety of food sold in the canteen or the lack of more interesting extracurricular activities.

Working in groups of three or four, select one issue that you want to work on, collect the relevant data and write a report to recommend how the school can improve the situation. The report should include:

- background of the problem,
- question posed for the statistical investigation,
- method of data collection, including choice of sample and design of test instrument,
- organisation and display of data using an appropriate statistical diagram drawn using a software,
- analysis and interpretation of the data,
- conclusion and recommendation.

Basic | Intermediate

Exercise

1. The table shows the number of students who travel to school by bus, by car, by bicycle and on foot respectively.

Mode of transport	Bus	Car	Bicycle	Foot
Number of students	768	256	64	192

Construct a pie chart using data from the table.

2. A survey is conducted to find out which of the four ice-cream flavours, chocolate, strawberry, mango and vanilla, the students in a class prefer. The pie chart shows the results of the survey.

(i) If one-quarter of the class prefers strawberry, state the angle of the sector that represents this information.

(ii) Find the angle of the sector that represents the number of students who prefer vanilla.

(iii) Express the number of students who prefer vanilla as a percentage of the total number of students in the class.

(iv) If 5 students prefer mango, find the total number of students in the class.

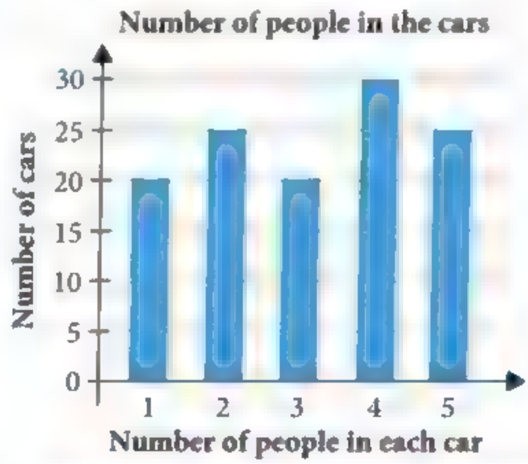
3. The pie chart shows the sources of revenue of a publishing company.

(i) Express the revenue the company earns from newspapers as a percentage of the total revenue of the company.

(ii) Express the revenue the company earns from magazines as a percentage of the total revenue of the company.

(iii) If the revenue the company earns from books is $17\frac{1}{2}\%$ of the total revenue of the company, find the value of x.

4. The bar graph illustrates the results of a survey conducted on cars at a traffic junction.



(i) Find the total number of cars in the survey.

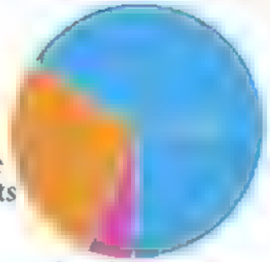
(ii) Find the total number of people in all the cars.

(iii) Express the number of cars with 4 or more people as a percentage of the total number of cars surveyed.

(iv) If the information is illustrated on a pie chart, find the angle of each sector.

Exercise 13B

5. The pie chart shows the distribution of students and teachers in a school.



   (i) If there are five times as many female students as teachers in the school, find the angle of the sector that represents the number of teachers in the school.

   (ii) If there are 45 teachers in the school, find the number of
   (a) female students, **(b)** male students,
   in the school.

   (iii) If $\frac{2}{3}$ of the teachers are women, express the number of females in the school as a percentage of the total school population.

6. A factory produces three products, A, B and C, in the ratio $1 : x : 5$. When this information is illustrated on a pie chart, the angle of the sector that represents the quantity of C produced is 120°. Find the value of x.

[illegible] The pie chart shows the market share of 4 different brands of floor detergents. Company X produces Brands A and B while Company Y produces Brands C and D. Company X claims that it has a bigger share of the market for floor detergents than Company Y.
Do you agree? Explain.

Statistics is concerned with the collection, organisation, analysis and interpretation of data to answer important questions in the real world. In this chapter, we have learnt how to pose statistical questions, use the statistical investigation cycle, and represent our findings using statistical diagrams. Diagrams allow us to visualise information efficiently but they can also be misleading when presented or used inappropriately.

Many of the questions we face today are statistical in nature. You may have heard about [illegible] or [illegible] as the fourth industrial revolution (or Industry 4.0) approaches. Statistics form the foundation for these new emerging fields, and we see that data can be a way for us to model or view the world. In the next few years, we will be learning more analytical methods as well as how to describe the features of the data at hand. The correct interpretation and usage of each statistical diagram is key to effectively communicate the answers to our questions.

1. **Stages of statistical investigation**

(a) Pose question (b) Collect data (c) Organise and display data (d) Analyse and interpret data

2. **Types of statistical data**

(a) **Categorical** data (b) **Numerical** data

- Give an example of categorical data and an example of numerical data. How are they different from each other?

3. **Types of statistical diagrams**

(a) A frequency table is primarily used to organise the data collected.

(b) A pictogram is appealing but not suitable for data that require a fraction of a picture drawn.

(c) A bar graph is used to compare the size of each category with one another.

(d) A pie chart is used to compare the relative size of each category with the whole.

- Give a key feature of each statistical diagram.

Answer Keys

Chapter 1 Primes, Highest Common Factor and Lowest Common Multiple

Practise Now 1
1. Composite; prime

Practise Now 2
1. 32
2. 29

Practise Now 3
1. $2 \times 3^2 \times 7$
2. $2^3 \times 3^2 \times 11$
3. (i) 43×47
   (ii) (1, 2021), (43, 47)

Practise Now 4
1. 3 cm by 5 cm by 13 cm
2. 18 cm

Practise Now 5
1. 28
2. 84

Practise Now 6
1. 14
2. 21
3. (i) 45 (ii) $p = 5$, $q = 3$

Practise Now 7
(a) 11 (b) 5

Practise Now 8
1. (a) 231 (b) 0.3582
2. 125.7 cm
3. 292 cubes

Practise Now 9
1. 28
2. 28

Practise Now 10
1. 360
2. 2520

Practise Now 11
1. 1, 3, 5, 9, 15, 27, 45, 135
2. 1, 2, 4, 5, 8, 10, 20, 40

Practise Now 12
9

Practise Now 13
1. 490 and 2205
2. (i) 21 (ii) 20

Practise Now 14
1. 5.00 p.m.
2. (i) 28 cm (ii) 11

Introductory Problem Revisited
(i) 16 cm (ii) 12

Exercise 1A
1. (a) Composite
   (b) Prime (c) Prime
   (d) Composite
2. 38
3. 43
4. (a) $2^3 \times 3^2$ (b) $2^2 \times 3^3 \times 7$
   (c) 11×17
   (d) $2 \times 3 \times 5 \times 7$
5. (i) 2×1013
   (ii) (1, 2026), (2, 1013)
6. 3 cm by 7 cm by 13 cm
7. (a) 2011, 2017
   (b) 2027, 2029
8. 2028
9. (a) $2^4 \times 7^2 \times 11$
   (b) $2^3 \times 3^3 \times 7$
   (c) $3^4 \times 17 \times 19$
   (d) $2^5 \times 5^3 \times 7^2$
10. (1, 2022), (2, 1011), (3, 674), (6, 337)
11. Any 3:
    2 cm by 3 cm by 35 cm
    2 cm by 5 cm by 21 cm
    2 cm by 7 cm by 15 cm
    3 cm by 5 cm by 14 cm
    3 cm by 7 cm by 10 cm
    5 cm by 6 cm by 7 cm
12. 21 cm or 24 cm

Exercise 1B
1. (a) 42 (b) 24
   (c) 54 (d) 56
2. (a) 99 (b) 189
   (c) 156 (d) 720
3. (a) 15 (b) 12
   (c) 18 (d) 20
4. (a) 28 (b) 36
   (c) 66 (d) 120
5. (a) 8 (b) 9
   (c) 6 (d) 9
6. (a) 9291 (b) 1.0024
   (c) 9
8. 36
9. 20 and 21
10. 63.2 cm
11. 169 cm^2
12. 84
14. (i) 18 (ii) $p = 2$, $q = 3$

Exercise 1C
1. (a) 6 (b) 13
   (c) 6 (d) 90
   (e) 1 (f) 1
2. 12
3. (a) 180 (b) 462
   (c) 2160 (d) 209
   (e) 864 (f) 13 728
5. 1, 2, 5, 10, 25, 50, 125, 250
6. 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200
7. 20
9. 30 096
10. 1, 2, 4, 8, 13, 26, 52, 104
11. 1400 and 1575
12. (i) $2 \times 3 \times 5^2 \times 7$
    (ii) 42 and 525
13. (i) 9 (ii) 19, 7
14. 3
15. (i) 240 seconds
    (ii) 20 minutes
16. (i) 5 cm (ii) 130
17. Possible numbers: 120, 168
18. (i) 45 (ii) 44
19. (i) 208 (ii) 208
20. (i) 34 (ii) 10

Chapter 2 Fractions

Practise Now 1
Simplest form $\frac{1}{2}, \frac{3}{4}, \frac{5}{7}, \frac{13}{23}$

$\frac{4}{6} = \frac{2}{3}$; $\frac{6}{9} = \frac{2}{3}$; $\frac{8}{10} = \frac{4}{5}$;

Not simplest form:

$\frac{20}{200} = \frac{1}{10}$

Practise Now 2
1. (a) < (b) >
   (c) = (d) >
2. $\frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{7}{9}, \frac{5}{6}$

Practise Now 3
(a) $\frac{8}{3}$ (b) $\frac{29}{5}$

Practise Now 4
(a) $6\frac{3}{4}$ (b) $3\frac{1}{3}$

Practise Now 5
(a) = (b) <
(c) >

Practise Now 6
1. (a) $\frac{8}{13}$ (b) $\frac{8}{9}$
   (c) $1\frac{7}{12}$ (d) $5\frac{1}{3}$
2. (a) $\frac{4}{15}$ (b) $1\frac{1}{4}$
   (c) $\frac{11}{20}$ (d) $1\frac{7}{36}$

Practise Now 7
1. (a) $3\frac{2}{3}$ (b) $8\frac{1}{11}$
   (c) $5\frac{23}{48}$ (d) $3\frac{2}{21}$
2. (a) $\frac{2}{3}$ (b) $8\frac{1}{7}$
   (c) $2\frac{7}{18}$ (d) $\frac{23}{30}$
3. $1\frac{49}{60}$ h

Practise Now 8
(a) 12 (b) $13\frac{1}{2}$
(c) $103\frac{1}{2}$

Practise Now 9
(a) $\frac{2}{15}$ (b) $\frac{1}{13}$
(c) $\frac{7}{9}$ (d) $3\frac{199}{315}$

Practise Now 10
(a) $\frac{1}{5}$ (b) 690

Practise Now 11
(a) $\frac{1}{12}$ (b) $\frac{1}{12}$
(c) $\frac{3}{14}$ (d) $\frac{11}{36}$

Practise Now 12
1. (a) 54 (b) $1\frac{1}{3}$
(c) $1\frac{1}{35}$ (d) $2\frac{2}{5}$
(e) $11\frac{1}{4}$ (f) $4\frac{1}{2}$
2. (i) $10\frac{3}{8}$ (ii) $14\frac{3}{8}$

Introductory Problem Revisited
(a) 96
(b) 2 cups of butter, 2 cups of brown sugar, 4 eggs, 4 teaspoons of vanilla extract, 2 teaspoons of baking soda, 6 cups of flour, 440 g of chocolate chips

Exercise 2A
1. (a) $\frac{1}{9}$ (b) $\frac{5}{6}$
(c) $\frac{2}{5}$ (d) $\frac{2}{3}$
2. (a) < (b) =
(c) > (d) <
3. (a) $\frac{38}{3}$ (b) $\frac{35}{8}$
(c) $\frac{17}{3}$ (d) $\frac{89}{9}$
4. (a) $17\frac{1}{2}$ (b) $1\frac{1}{2}$
(c) $8\frac{1}{3}$ (d) $1\frac{2}{3}$
5. (a) = (b) <
(c) = (d) >
(e) < (f) =
(g) = (h) >
6. (a) $\frac{13}{17}$ (b) $1\frac{1}{5}$
(c) $1\frac{2}{9}$ (d) $1\frac{1}{9}$
(e) $1\frac{7}{12}$ (f) $\frac{3}{13}$
(g) $\frac{9}{20}$ (h) $2\frac{1}{3}$
(i) $\frac{1}{10}$ (j) $\frac{11}{21}$
7. (a) $3\frac{6}{7}$ (b) $2\frac{1}{9}$
(c) 8 (d) $7\frac{7}{10}$
(e) $4\frac{2}{3}$ (f) $2\frac{2}{15}$
(g) $5\frac{1}{12}$ (h) $4\frac{41}{77}$
8. (a) $7\frac{2}{5}$ (b) $1\frac{6}{7}$
(c) $\frac{2}{3}$ (d) $\frac{7}{22}$
(e) $3\frac{5}{6}$ (f) $1\frac{7}{12}$
(g) $1\frac{19}{24}$ (h) $2\frac{11}{35}$
9. $\frac{1}{6}, \frac{5}{12}, \frac{4}{9}, \frac{11}{18}, \frac{2}{3}$
10. (i) $\frac{19}{5}, \frac{47}{15}, \frac{16}{5}, \frac{11}{3}, \frac{18}{5}$
(ii) $\frac{47}{15}, \frac{32}{10}, 3\frac{15}{25}, 3\frac{2}{3}, 3\frac{4}{5}$
11. (a) $1\frac{7}{16}$ (b) $\frac{2}{9}$
(c) $1\frac{1}{12}$ (d) $5\frac{5}{12}$
(e) $12\frac{2}{7}$ (f) $5\frac{9}{70}$
(g) $\frac{7}{18}$ (h) $\frac{65}{84}$
(i) $2\frac{13}{16}$ (j) $8\frac{1}{8}$
12. $1\frac{13}{20}$ l
13. $\frac{3}{20}$ kg
14. $5\frac{3}{10}$ km
15. 9
16. 4

Exercise 2B
1. (a) 16 (b) 105
(c) $94\frac{1}{2}$ (d) 49
(e) $\frac{1}{10}$ (f) $\frac{11}{40}$
(g) $\frac{25}{48}$ (h) $\frac{10}{33}$
(i) $1\frac{1}{2}$ (j) $2\frac{1}{2}$
(k) 1 (l) $2\frac{11}{12}$
2. (a) $\frac{1}{6}$ (b) $\frac{2}{21}$
(c) $2\frac{2}{5}$ (d) $\frac{31}{52}$
(e) 10 (f) $27\frac{1}{2}$
(g) $\frac{5}{6}$ (h) $\frac{5}{12}$
(i) $1\frac{1}{2}$ (j) $1\frac{1}{3}$
(k) $1\frac{2}{81}$ (l) $1\frac{1}{9}$
3. $4\frac{4}{7}$ h
4. (i) $\frac{1}{10}$ (ii) 135
5. $\frac{27}{160}$ kg
6. \$48
7. 1344
8. $31\frac{1}{5}$ cm^3
9. PKR 56 000
10. 8
11. PKR 420

Chapter 3 Decimals

Practise Now 1
1. (a) $\frac{1}{20}$ (b) $\frac{13}{100}$
(c) $2\frac{1}{25}$ (d) $7\frac{177}{500}$
2. (a) 0.4 (b) 0.45
(c) 0.312 (d) 1.05
(e) 2.75
3. (a) > (b) =
(c) < (d) <

Introductory Problem Revisited
1. (a) 0.5 (b) 0.375
(c) 0.3333 (d) 0.1667
(e) 0.142 857 1429
2. (a) $\frac{3}{10}$ (b) $\frac{2}{5}$
(c) $\frac{1}{4}$ (d) $\frac{167}{1000}$
(e) $\frac{5}{8}$

Practise Now 2A
1. (a) $4.\dot{4}$ (b) $15.\dot{3}1\dot{0}$
(c) $20.1\dot{6}\dot{4}$
2. (a) 0.474 747
(b) 0.023 232
(c) 1.203 203

Practise Now 2B
(a) $0.\dot{2}$ (b) $0.8\dot{3}$
(c) $0.\dot{0}\dot{9}$ (d) $0.\dot{4}28\,57\dot{1}$

Practise Now 3
(a) $\frac{7}{9}$ (b) $\frac{36}{99}$
(c) $\frac{5}{99}$ (d) $\frac{167}{999}$

Practise Now 4
(a) $\frac{5}{6}$ (b) $\frac{3}{22}$
(c) $\frac{5}{12}$ (d) $\frac{70}{111}$

Practise Now 5
(a) $1\frac{56}{99}$ (b) $2\frac{61}{90}$
(c) $2\frac{214}{495}$ (d) $3\frac{1241}{4995}$

Practise Now 6A
1. (a) 11.13 (b) 11.94
(c) 31.744 (d) 36.11
2. (a) 2.04 (b) 1.6
(c) 35.25 (d) 24.16

Practise Now 6B
(a) 10.5 (b) 5.44
(c) 619.20

Practise Now 7
(a) 32.544 (b) 48.3678

Practise Now 8
(a) 0.05 (b) 0.675
(c) 0.26

Practise Now 9
(a) 2.3 (b) 12.3

Practise Now 10A
1. (a) 7 (b) 6.3
(c) 3610 (d) 32 900
2. (a) 0.09 (b) 1.37
(c) 0.1431 (d) 0.0027

Practise Now 10B
(a) 3608 m (b) 705 500 cm
(c) 1.385 km (d) 0.485 m

Practise Now 11

(a) 57.5 kg (b) 3 040 000 g
(c) 6.975 tonnes
(d) 0.077 542 tonnes

Practise Now 12

(a) 950 ml (b) 5 423 050 ml
(c) 2.765 *l* (d) 0.089 02 *l*

Exercise 3A

1. (a) $\frac{1}{2}$ (b) $\frac{47}{100}$
   (c) $\frac{9}{25}$ (d) $\frac{1}{20}$
   (e) $2\frac{1}{25}$ (f) $\frac{157}{200}$
   (g) $7\frac{71}{200}$ (h) $1\frac{1}{25}$
2. (a) 0.6 (b) 0.15
   (c) 0.3 (d) 0.92
   (e) 3.5 (f) 5.75
   (g) 6.25 (h) 5.625
3. (a) $>$ (b) $<$
   (c) $=$ (d) $<$
   (e) $>$ (f) $=$
   (g) $<$ (h) $<$
4. (a) $0.\dot{2}$ (b) $0.\dot{1}2\dot{3}$
   (c) $0.1\dot{7}$ (d) $0.23\dot{4}\dot{2}$
5. (a) 0.121 212
   (b) 0.711 111
   (c) 0.353 353
   (d) 0.332 242
6. (a) $0.\dot{7}$ (b) $0.0\dot{5}$
   (c) $0.7\dot{3}$ (d) $0.41\dot{6}$
7. (a) $\frac{5}{9}$ (b) $\frac{2}{3}$
   (c) $\frac{29}{33}$ (d) $\frac{121}{999}$
   (e) $\frac{11}{90}$ (f) $\frac{7}{90}$
   (g) $\frac{767}{990}$ (h) $\frac{697}{900}$
8. (a) $1\frac{2}{3}$ (b) $2\frac{[illegible]}{99}$
   (c) $3\frac{415}{999}$ (d) $1\frac{26}{45}$
9. (a) $1.0\dot{1}7\dot{3}$ (b) $4.\dot{4}$
   (c) $20.02\dot{2}\dot{3}$ (d) $7.\dot{1}47\,56\dot{3}$
10. (a) 0.010 210 210
    (b) 12.111 211 211
    (c) 10.003 333 333
    (d) 3.033 203 320
11. (a) $0.\dot{1}\dot{8}$ (b) $0.1\dot{3}\dot{6}$
    (c) $0.\dot{5}71\,42\dot{8}$ (d) $0.0\dot{7}6\,923\dot{0}$
12. (a) $\frac{4}{33}$ (b) $\frac{8}{11}$
    (c) $\frac{6}{37}$ (d) $\frac{8}{101}$
    (e) $\frac{7}{45}$ (f) $\frac{1}{450}$
    (g) $\frac{1}{495}$ (h) $\frac{49}{300}$
13. (a) $1\frac{7}{90}$ (b) $2\frac{26}{225}$
    (c) $2\frac{251}{495}$
14. (a) $0.\dot{0}58\,823\,529\,411\,764\dot{7}$
    (b) $0.\dot{0}52\,631\,578\,947\,368\,42\dot{1}$
15. (a) $\frac{5}{13}$ (b) $\frac{3}{7}$
    (c) $\frac{1879}{9900}$ (d) $\frac{316}{1665}$
16. (a) $1\frac{359}{3300}$ (b) $2\frac{1222}{4995}$
17. 0

Exercise 3B

1. (a) 2.7 (b) 2.02
   (c) 1.594 (d) 3.09
   (e) 12.18 (f) 1.604
2. (a) 1.3 (b) 7.38
   (c) 8.213 (d) 0.8
   (e) 3.13 (f) 6.88
3. (a) 117.76 (b) 0.441
   (c) 76.692 (d) 0.18
4. (a) 17.664 (b) 19.56
   (c) 0.0216 (d) 0.490 75
5. (a) 0.27 (b) 0.8565
   (c) 0.033 25 (d) 0.218 25
6. (a) 2.7 (b) 11
   (c) 57.1 (d) 2.9
7. (a) 6.3 (b) 4012.5
   (c) 25.1 (d) 1.36
   (e) 5.305 (f) 0.026 68
8. (a) 123 m (b) 8300 cm
   (c) 1 556 mm (d) 10.37 m
   (e) 0.503 km (f) 0.25 cm
9. (a) 6230 kg (b) 66 g
   (c) 25.6 kg (d) 0.365 kg
   (e) 89.234 tonnes
   (f) 0.002 056 tonnes
10. (a) 2546 ml
    (b) 45 000 ml
    (c) 8.926 *l*
    (d) 0.003 02 *l*
11. \$85
12. (a) 6 m 15 cm
    (b) 6.055 km
    (c) 54 440 mm
    (d) 0.004 6223 km
    (e) 0.896 m
    (f) 9 kg 123 g
    (g) 10.365 kg
    (h) 42 003 g
13. 1022.96 m
14. PKR 636
15. 346 ml
16. \$205.40
17. 1078

Chapter 4 Integers, Rational Numbers and Real Numbers

Practise Now 1A

1. (i) 2020, 6 (ii) 5, 17
   (iii) 2020, 1.666, $\frac{3}{4}$, 6
   (iv) −5, $-\frac{1}{2}$, −3.8, −17, $-\frac{5}{3}$
2. (a) −43.6 °C (b) −10 994 m
   (c) −\$10 000 (d) −81°

Practise Now 1B

1. (b) $<$ (c) $>$
   (d) $>$
2. −5, −3.8, $-1\frac{1}{2}$, 0, $\frac{3}{4}$, 1.666, 4

Practise Now 1C

(a) −9 (b) −16
(c) −94 (d) −115

Practise Now 2

(a) 5 (b) 28
(c) 4 (d) 23

Practise Now 3

(a) −4 (b) −21
(c) 3 (d) 45

Practise Now 4

(a) −20 (b) −51
(c) −364

Practise Now 5

(a) 10 (b) 46
(c) −5 (d) 18

Practise Now 6

1. 28 °C
2. −165 m; 479 m

Introductory Problem Revisited

(i) 10 000 m (ii) Mauna Kea

Practise Now 7A

(a) −18 (b) −32
(c) 42 (d) 10
(e) −33 (f) −190
(g) 36 (h) 28
(i) −2020

Practise Now 7B

(a) −4 (b) −5
(c) 3 (d) −4
(e) 4 (f) 8

Practise Now 7C

1. (a) ±1, ±2, ±4, ±8
   (b) ±1, ±3, ±9
   (c) ±1, ±7 (d) ±1
2. (a) 4, −4 (b) 12, −12
   (c) 5, −5 (d) 6, −6

Practise Now 8A

(a) ±8 (b) −3
(c) 6

Practise Now 8B

(a) −27 (b) 4
(c) −2 (d) 3

Practise Now 9A

(a) −32 (b) −55

Practise Now 10

(a) $3\frac{9}{10}$ (b) $3\frac{7}{12}$
(c) $-\frac{11}{12}$ (d) $-\frac{5}{18}$

Practise Now 11

(a) $-\frac{1}{2}$ (b) $\frac{29}{38}$
(c) $-1\frac{7}{8}$ (d) $-2\frac{2}{5}$

Practise Now 12A

(a) −8 (b) $-1\frac{1}{7}$
(c) $-21\frac{1}{8}$ (d) $\frac{3}{8}$

Practise Now 13

(a) 101 036 (b) 9.6
(c) 140

Practise Now 14

(a) −0.855 (b) 3.823 92
(c) −26.25 (d) −48.1016

Practise Now 15

(a) −0.115 (b) 34.56
(c) −0.5 (d) −2

Practise Now 16

0.583

Exercise 4A

1. (i) 10 001, 4 (ii) −12, −2026
   (iii) $\frac{1}{5}$, 4.33, 10 001, 4
   (iv) −0.3, $\frac{9}{7}$, 12, $1\frac{1}{2}$, 2026
2. (a) 273.15 °C
   (b) −86 m (c) −$85 000
   (d) −0.15 kg
3. (a) < (b) <
   (c) < (d) >
   (e) < (f) >
5. (a) −13, −3, 23, 30, 230
   (b) −10, −0.5, $-\frac{3}{20}$, 15, 150
6. (a) 10 m below sea level
   (b) A withdrawal of $25
   (c) An anticlockwise rotation of 30°
   (d) A deduction of 4 marks
7. (a) > (b) >
   (c) < (d) >
9. (a) −101, −102
   (b) 0, 1, 2 (c) −7, −6, −1, 0
   (d) −4.9, −4.5
10. False
11. Liquid nitrogen

Exercise 4B

1. (a) −11 (b) −7
   (c) −11 (d) −7
2. (a) 5 (b) 7
   (c) 12 (d) 0
3. (a) −5 (b) −11
   (c) −7 (d) −3
   (e) −19 (f) −5
4. (a) −9 (b) −13
   (c) −13 (d) −9
5. (a) 12 (b) 11
   (c) −6 (d) 3
6. 23 °C
7. −2 °C
8. HOLIDAY
9. (a) −30 (b) −79
   (c) −104 (d) −160
10. (a) 34 (b) 113
    (c) 18 (d) 61
11. (a) −53 (b) −62
    (c) −35 (d) −107
    (e) −47 (f) −99
12. (a) −107 (b) −145
    (c) −1430 (d) −1464
13. (a) 35 (b) 95
    (c) 41 (d) 57
14. 51 m; 189 m
15. (a) Negative (b) Negative
    (c) Negative (d) Positive
16. (i) 5 (iii) 4 years
17. (a) Examples: $x = -3, y = -7$; $x = 2, y = -12$
    (b) Examples: $x = -3, y = 7$; $x = 2, y = 12$

Exercise 4C

1. (a) −35 (b) −24
   (c) 36 (d) 716
   (e) 697 (f) 88
   (g) 0 (h) −84
2. (a) −3 (b) −8
   (c) 2 (d) −2
   (e) −9 (f) 6
3. (a) ±1, ±2, ±3, ±4, ±6, ±12
   (b) ±1, ±23
   (c) ±1, ±2, ±4, ±8, ±16
   (d) ±1
4. (a) 10, −10 (b) 24, −24
   (c) 34, −34 (d) Not possible
5. (a) ±9 (b) ±4
   (c) ±5 (d) ±10
6. (a) 9 (b) 2
   (c) −3 (d) Not possible
7. (a) −64 (b) −64
   (c) 64 (d) 2
   (e) −5 (f) −6
   (g) 4 (h) −10
8. (a) 77 (b) 3
   (c) −130 (d) 43
   (e) 21 (f) 5
   (g) 2 (h) −80
   (i) −2 (j) 4
10. Negative
11. 1, −1
12. No
13. ±1, ±3
14. (a) 40 (b) 105
    (c) −76 (d) 14
    (e) 64 (f) $51\frac{1}{4}$
    (g) $1\frac{1}{3}$ (h) 5
15. 46 marks
17. 4 points
18. (i) 8 questions
    (ii) Example: 3 correct, 14 wrong
19. $(h - k)$ units
20. 8, −4

Exercise 4D

1. (a) $-1\frac{1}{4}$ (b) $-2\frac{3}{10}$
   (c) $2\frac{7}{8}$ (d) $8\frac{5}{6}$
2. (a) $-1\frac{3}{5}$ (b) $-\frac{2}{25}$
   (c) $-1\frac{5}{16}$ (d) −11
   (e) $-3\frac{1}{3}$ (f) $\frac{8}{15}$
4. (a) −7.8 (b) −4.3
   (c) 2.2 (d) −23.1
   (e) −18.1 (f) −43.2
   (g) 14.1 (h) 0.8
5. (a) 19.136 (b) 0.843 18
   (c) −1.222 (d) −57.1
   (e) −11 (f) −1.28
6. (a) $-1\frac{2}{3}$ (b) $-4\frac{3}{20}$
   (c) $-12\frac{11}{24}$ (d) $1\frac{1}{20}$
   (e) $-\frac{17}{24}$ (f) $6\frac{1}{12}$
7. (a) $\frac{1}{7}$ (b) −1
   (c) $9\frac{1}{8}$ (d) $-5\frac{1}{10}$
9. (a) $-8\frac{3}{16}$ (b) $2\frac{4}{9}$
   (c) $-1\frac{1}{4}$ (d) 1
10. (a) −105.04 (b) −9.27
    (c) −11.387 (d) −5.349
    (e) 129.07 (f) −21.73
    (g) 62.858 (h) 23.933
11. (a) −0.04 (b) −7
    (c) −0.36 (d) −0.03
    (e) 2.6 (f) 23.3275
13. (a) 16.934 (b) −2.085
    (c) −5.842 (d) 7.288
14. 0.135 791 113…, 0.369 121 518…

Chapter 5 Approximation and Estimation

Practise Now 1

1. (a) 3 409 730
   (b) 3 409 700
   (c) 3 410 000
   (d) 3 410 000
2. 19 149 999; 19 050 000

Practise Now 2A

1. (a) 78.5 (b) 78
   (c) 78.47 (d) 78.470
2. No

Practise Now 2B

(a) 3 (b) 5
(c) 1 (d) 2

Practise Now 2C

(a) 4 (b) 3
(c) 5 (d) 6

Practise Now 2D

(a) 2 (b) 3
(c) 3 (d) 5

Practise Now 2E

1. (a) 2 (b) 3
   (c) 6 (d) 4
2. 4.10 cm

Practise Now 2F

(a) 3 (b) 5
(c) 2

Practise Now 3

1. (a) 3750 (b) 0.004 710
   (c) 5000 (d) 0.10, 0.100
2. 21 249; 21 150

Practise Now 4A

(a) $1.55 \leqslant a < 1.65$
(b) $0.55 \leqslant b < 0.65$
(c) $1.25 \leqslant c < 1.35$

Practise Now 4B

1. 13.2 km $\leqslant$ distance between A and B < 13.5 km
2. 220

Practise Now 5

(i) 10.2 cm (ii) 41.0 cm

Practise Now 6

1. Not reasonable
2. (a) 19; 19.2
   (b) 40; 40.1
3. $\frac{240}{80}$ hours

Practise Now 7

PKR 475

Practise Now 8

Option A

Practise Now 9

$66\frac{2}{3}\%$

Exercise 5A

1. (a) 698 350 (b) 698 400
   (c) 698 000 (d) 700 000
2. (a) 44 970 (b) 45 000
   (c) 45 000 (d) 40 000
3. (a) 45.7 (b) 46
   (c) 45.74 (d) 45.740
4. (a) 8 (b) 7.70
   (c) 7.7 (d) 7.697 15
5. (a) 5 (b) 5
   (c) 8 (d) 3
   (e) 2 (f) 4
6. (a) 730 (b) 504.0
   (c) 0.003 020 (d) 6400, 6400
   (e) 10.0 (f) 8.00
7. (a) $45.5 \leqslant a < 46.5$
   (b) $12.55 \leqslant b < 12.65$
   (c) $0.395 \leqslant c < 0.405$
   (d) $2945 \leqslant d < 2955$
   (e) $295 \leqslant e < 305$
   (f) $4495 \leqslant f < 4505$
8. (a) 26.4, 16.3
   (b) 23.8, 13.7
   (c) 2, 1.83
   (d) 17.1, 10.1
9. 235 824 999, 235 815 000
10. (a) 4.9 m (b) 10 cm
    (c) \$11.00 (d) 6.49 kg
11. Possible values: 47.44 s, 47.40 s, 47.38 s
12. 3, 4, 5, 6
13. 320 499 m^2, 319 500 m^2
14. Possible values: 0.1195 cm, 0.1201 cm, 0.120 08 cm
15. 6
16. 3723.875 cm^3 $\leqslant$ volume < 4492.125 cm^3
17. (i) 20.7 m^2 $\leqslant$ area < 35.2425 m^2
    (ii) 3438
18. No
19. No
21. 14.8, 14.3
22. 24.8 cm

Exercise 5B

1. 6
2. 2 m
4. (i) 16.2 cm (ii) 65.0 cm
5. (i) 2.13 m (ii) 28.3 m
6. Not reasonable
7. (a) 80 000, 78 507
   (b) 3, 2.99
8. (i) 3.6; 30 (ii) 0.12
   (iii) 0.121 (iv) Yes
9. 1 : 2
10. (i) 546.4784 m^2
    (ii) \$383
11. $\frac{270}{9}$ l
12. \$3500
13. PKR 1600
14. Option A
15. (c) 1500 kg
16. 14 m
17. PKR 6000
18. Shop B

Chapter 6 Basic Algebra and Algebraic Manipulation

Practise Now 1

1. (a) 28 (b) $-1\frac{1}{2}$
2. $12\frac{1}{4}$

Practise Now 2

(a) $-4x$ (b) $-14x$
(c) $4x$ (d) $23x$
(e) $20y$ (f) $51y$
(g) $-5z$ (h) $18z$

Practise Now 3

(a) $6x - 4y$ (b) $6y - 7x$
(c) $8x + 3z + 3$
(d) $7a - 10b + 13$

Practise Now 4

(a) $2x - 14$ (b) $20y - 15x$
(c) $8 + ax - 2az$ (d) $6 + 3by + bz$

Practise Now 5

(a) $26x + 4y$ (b) $7x - 4y$
(c) $5x - 8$

Practise Now 6

(a) $5(2x + 5)$ (b) $6(3 - 2x)$
(c) $7a(3 - 2y)$
(d) $3a(11x + 9 + y)$
(e) $18x(1 - 3y + 2z)$
(f) $10x(3y - 1 - 5z)$

Practise Now 7

(a) $5(h + 5)$ (b) $7a(2y + 3)$
(c) $-3z(3 + 8b + 5c)$
(d) $-4(1 + 4x + 5xy)$

Practise Now 8

(a) $\frac{1}{6}x - \frac{3}{20}y$
(b) $y - 6x$

Practise Now 9

1. (a) $\frac{7x - 19}{6}$ (b) $\frac{22 - x}{12}$
2. (a) $\frac{11 - 2x}{12}$ (b) $\frac{13x + 11}{9}$

Exercise 6A

1. (a) $ab + 5y$ (b) $f^2 - 3$
   (c) $6kq$ (d) $\frac{2w}{3xy}$
   (e) $3x - 4\sqrt{z}$ (f) $\frac{2p}{5q}$
2. (a) 52 (b) $3\frac{1}{2}$
   (c) 136 (d) $\frac{1}{2}$
3. (a) 69 (b) 57
   (c) $\frac{7}{15}$ (d) $1\frac{7}{15}$
4. (a) $3x$ (b) $5x$
   (c) $-6x$ (d) $-22x$
   (e) $-13y$ (f) $-19y$
   (g) $-5z$ (h) $13z$
5. (a) $-x - 1$ (b) $7x + 7y$
   (c) $4xy + 8x$ (d) $-2x + 5y - 4z$
6. (a) $2x - y$ (b) $-10a + 6b$
   (c) $-11c + 12d$
   (d) $8hk$
7. (a) $(p + q) - \sqrt[3]{3hk}$
   (b) $(20x + 500y)$ cents
8. (a) $2\frac{1}{4}$ (b) $2\frac{1}{10}$
   (c) $\frac{3}{8}$ (d) $\frac{6}{275}$
9. (a) $113x$ (b) $-107x$
   (c) $-47x$ (d) $18x$
   (e) $-107y$ (f) $-1430y$
   (g) $-11z$ (h) $157z$
10. (a) $-3a - 3b$ (b) $-10h + 5k$
    (c) $p + 14q$ (d) $-5x + 9y$
    (e) $22x + 3y - 33$
    (f) $3y - 14z + 10yz + 9$
11. (i) $11p - 2q - 5r$
    (ii) 50
12. (i) $(12m + 5)$ years
    (ii) $(15m + 10)$ years
13. \$$(11w + 12m)$
14. (a) $\frac{5}{2}a$ (b) $\frac{2}{5}b$
    (c) $\frac{2}{7}c$

Exercise 6B

1. (a) $x \quad 5$ (b) $4 + x$
   (c) $6y + 14$ (d) $16y - 40$
   (e) $32b - 24a$
   (f) $2ax - 2ay$
   (g) $5 + 6bw - 2bx$
   (h) $13 - 3cy \quad 9cz$
2. $(4x + 20)$ years
3. (a) $5a + 7b$ (b) $19p + 84q$
   (c) $7b - 4a$ (d) $7x - 9y + 10z$
4. $(10x - 6y)$ cents
5. (a) $3(4x + 3)$
   (b) $9b(3 - 4y)$
   (c) $2p(3r - 8q)$
   (d) $4a(2x + 3 - z)$
   (e) $6h(5k - 1 - 4n)$
   (f) $2m(9z \quad 2x \quad 3y)$
6. (a) $-5(5y + 7)$
   (b) $-16a(4x + y)$
   (c) $-9w(9 + x + 2y)$
   (d) $-12(2x + 3 + vz)$
7. (a) $8x - 2$
   (b) $4x - y - 4z$
   (c) $-12p - 13q + 20rs$
   (d) $2a + 2b - 9c - 4d$
8. (a) $15v - 2u$
   (b) $3b - 5a$
   (c) $11m - 8n$
   (d) $13x + 38$
   (e) $27b - 9a$
   (f) $9p - 14q$
   (g) $9y - 5x - 6$
   (h) $6q - 5p + 20$
   (i) $18a + 60b - 52c$
   (j) $-24x - 28y$
9. (a) $-6a - 16$
   (b) $25c + 5d$
10. (a) $-5x^2 + 6x + 13$
    (b) $16x^3 - 7x^2 - 8x + 29$
    (c) $-4w^2 - 3w - 17$
    (d) $18w^2 + 35w + 3$
11. (a) $5x(1 + 2b + 2c)$
    (b) $3x(-y + 2z)$
    (c) $-2x(6y + 7)$
    (d) $-13b(3b + a)$
    (e) $-3a(7b + 8)$
    (f) $-4axy(y + 4)^2$

Exercise 6C

1. (a) $\frac{1}{12}x + \frac{1}{10}y$
   (b) $\frac{8}{3}a - \frac{26}{35}b$
   (c) $-\frac{23}{72}c - \frac{7}{12}d$
   (d) $\frac{3}{2}f - \frac{5}{12}h - \frac{67}{20}k$
2. (a) $3a + \frac{11}{2}b \quad \frac{9}{2}c$
   (b) $2x \quad 3$ (c) $4p \quad 2$
   (d) $16x + 2$
3. (a) $\frac{9}{10}x$ (b) $\frac{1}{12}a$
   (c) $\frac{17h+7}{35}$ (d) $\frac{x-4}{8}$
   (e) $\frac{23x-3}{10}$ (f) $\frac{5y+3}{12}$
   (g) $\frac{a-11}{8}$ (h) $\frac{7}{12}q$
4. (a) $3y \quad 6x$ (b) $4q \quad 3p$
5. (a) $\frac{41x+13}{6}$ (b) $\frac{7-9x}{10}$
   (c) $\frac{18-17z}{20}$ (d) $\frac{-14p-25q}{6}$
   (e) $\frac{8b-9a}{15}$ (f) $\frac{23x-6}{20}$
   (g) $-\frac{13}{12}a$ (h) $1\frac{2}{3}$
   (i) $\frac{9}{14}a$ (j) $\frac{17x+32}{6}$
6. (a) $\frac{32p-41q}{14}$
   (b) $\frac{173b-41a}{30}$
   (c) $\frac{-21f-23h+16k}{12}$
   (d) $\frac{96+7x \quad 10y \quad 27z}{24}$

Chapter 7 Linear Equations

Practise Now 1

(a) 2 (b) $-4\frac{1}{2}$
(c) $-\frac{2}{5}$ (d) 3

Practise Now 2

1. (a) $5\frac{5}{6}$ (b) $-\frac{5}{7}$
2. (a) 1.2 (b) 2.05

Practise Now 3

(a) $2\frac{1}{2}$ (b) -18

Practise Now 4

1. 4 and 20
2. 15

Introductory Problem Revisited

9 cousins

Practise Now 5

$16\frac{1}{2}$

Practise Now 6

1. (a) 50 N (b) 1000 kg
2. -123

Practise Now 7

(i) $S = 4n + 12$ (ii) 68

Exercise 7A

1. (a) 7 (b) -14
   (c) 22 (d) 4
   (e) -7 (f) 6
   (g) 5 (h) 3
   (i) $\frac{6}{7}$ (j) $2\frac{1}{7}$
2. (a) 1 (b) -17
   (c) -4 (d) $3\frac{1}{5}$
3. (a) 1 (b) 4
   (c) $2\frac{4}{5}$ (d) -3
   (e) 2 (f) $\frac{2}{21}$
   (g) 38 (h) $-13\frac{1}{2}$
   (i) 13 (j) $-1\frac{6}{7}$
4. (a) $\frac{13}{28}$ (b) $\frac{5}{8}$
   (c) 21 (d) -8
   (e) 3 (f) 24
   (g) 4 (h) 10
5. (a) 6 (b) 1.2
   (c) 0.6 (d) 2.05
   (e) -3.7 (f) 2.25
   (g) $-\frac{11}{15}$ (h) -10
6. (a) 9 (b) 5
   (c) $2\frac{16}{25}$ (d) $-1\frac{1}{32}$
7. (a) $2\frac{1}{2}$ (b) 19
8. (a) -2 (b) $-5\frac{2}{3}$
   (c) $\frac{8}{19}$ (d) $-\frac{10}{13}$
   (e) 4
9. (a) 1 (b) $\frac{9}{10}$
   (c) 1 (d) $\frac{1}{13}$

(e) $47\frac{1}{2}$ (f) $2\frac{5}{12}$

10. (a) 4 (b) $-1\frac{1}{2}$
    (c) $-\frac{1}{2}$ (d) $-2\frac{2}{7}$
    (e) $2\frac{4}{9}$ (f) $\frac{1}{8}$
11. It is the solution
12. (a) 3 (b) $1\frac{7}{8}$
    (c) $7\frac{11}{26}$ (d) $-1\frac{1}{3}$
    (e) 50 (f) $-13\frac{1}{2}$
    (g) 18 (h) $-\frac{1}{3}$
13. $\frac{3}{4}$
14. $-\frac{8}{9}$

Exercise 7B

1. 8700 kg
2. 17
3. 19, 15, 13
4. 14
5. 28
6. 18
7. 8
8. 12
9. 25 years
10. \$12.80
11. 40
12. 7.83 km/h
13. 28
14. $\frac{9}{14}$
15. 25

Exercise 7C

1. $33\frac{1}{5}$
2. (i) 1386 (ii) 7
3. $1\frac{2}{5}$
4. (a) $P = xyz$ (b) $S = p^2 + q^2$
   (c) $A = \frac{m+n+p+q}{4}$
   (d) $T = 60a + b$
5. 3
6. $3\frac{1}{4}$
7. $24\frac{1}{20}$
8. $\frac{1}{3}$

9. $-6\frac{1}{27}$

10. $-\frac{18}{49}$

11. (i) $S = 4n + 12$

(ii) -80

12. (i) $S = 5n + 20$

(ii) 45

13. (i) $T = cd + \frac{ef}{100}$

(ii) 577.50

14. (i) 56.7 °C

(ii) Less common

(iii) −79.8 °F

Chapter 8 Percentage

Practise Now 1A

1. (a) $\frac{19}{25}$, 0.76

(b) $\frac{9}{100}$, 0.09

2. (a) 65% (b) 25%

Practise Now 1B

1. (a) $\frac{563}{600}$, 0.938

(b) $\frac{147}{1000}$, 0.147

2. (a) $57\frac{1}{7}$% or 57.1%

(b) 54.32%

Practise Now 2

1. (a) $3\frac{9}{50}$, 3.18

(b) $4\frac{29}{400}$, 4.0725

(c) $\frac{1}{500}$, 0.002

(d) $\frac{33}{50\,000}$, 0.000 66

2. (a) $4933\frac{1}{3}$% (b) 546.8%

(c) 0.16% (d) 0.85%

Practise Now 3

1. (a) 35 (b) 8

2. (a) 0.75 cm (b) 100.8 kg

(c) $115

Practise Now 5

(i) 37.5% (ii) 62.5%

Practise Now 6

1. (i) 64% (ii) 56.25%

2. $66\frac{2}{3}$% or 66.7%

Practise Now 8

Village B

Practise Now 9

Raspberry

Practise Now 10

(i) $3.21 (ii) 14.6%

Practise Now 11

$4\frac{2}{13}$% or 4.15% decrease

Practise Now 12

1. $6\frac{2}{3}$% or 6.67% increase

2. 1% decrease

Practise Now 13

50

Practise Now 14

1. PKR 130 000

2. $125 000

Practise Now 15

1. PKR 745 032

2. $120 000

Exercise 8A

1. (a) $\frac{47}{100}$, 0.47

(b) $\frac{16}{25}$, 0.64

2. (a) 80% (b) 38%

3. (a) $\frac{41}{800}$, 0.051 25

(b) $\frac{457}{500}$, 0.914

4. (a) $66\frac{2}{3}$% or 66.7%

(b) 77.6%

5. (a) $1\frac{59}{100}$, 1.59

(b) $8\frac{67}{500}$, 8.134

(c) $\frac{3}{1000}$, 0.003

(d) $\frac{7}{50\,000}$, 0.000 14

6. (a) $8166\frac{2}{3}$% (b) 912.4%

(c) 0.23% (d) 0.275%

7. 84

8. 5610

9. (a) 0.036 m (b) 26.832 kg

(c) $218.75

11. 48.9%

12. (a) 75%

(b) $11\frac{19}{21}$% or 11.9%

(c) 300% (d) 1.5%

(e) 67% (f) 125%

(g) $16\frac{2}{3}$% or 16.7%

(h) 30%

13. Waseem

14. 7

16. 8.61%

17. 45%

18. Vasi, gold; David, silver; Cheryl, bronze

19. $33\frac{1}{3}$% or 33.3%

20. (i) 30%

21. Applicant C

22. 20%

23. 50

25. (ii) Possible incomes: $38 400 in 2022, $32 000 in 2023

Exercise 8B

1. (a) 81 (b) 63.196

(c) 66 (d) 135

2. 25%

3. 25%

4. $452 880

5. $86 400

6. 300

7. (a) 85 (b) 28

(c) 140 (d) 240

8. 16.4 million

9. PKR 14 500

10. 2496

11. 16.64%

12. 3% increase

14. $3\frac{1}{3}$% or 3.33% decrease

15. 1% decrease

16. $680 000

17. PKR 4 480 000

18. $64 000

19. 120%

20. (a) 35% decrease

(b) 15% increase

21. (a) Scheme A

(b) Possible salary: PKR 200 000

Chapter 9 Ratio and Rate

Practise Now 1

(i) 33 : 20 (ii) 20 : 53

Practise Now 2

(a) 9 : 5 (b) 3 : 10

Practise Now 3

10 : 3

Practise Now 4

1. (a) $3\frac{11}{15}$ (b) $3\frac{1}{5}$

2. 6 : 1

Practise Now 5

1. 609

2. $900

Practise Now 6

1. (i) 10 : 12 : 27

(ii) 10 : 27

2. 2 : 3 : 1

Practise Now 7

$360

Practise Now 8

(i) 2 (ii) 1 : 9

Practise Now 9

Vasi

Practise Now 11

1.45-kg tin

Practise Now 12

1. (a) $2.44

(b) (i) 614.8 km

(ii) $304

2. 60

Practise Now 13

1. (a) (i) PKR 421 856

(ii) PKR 137 114

(b) (i) EUR 17.51

(ii) THB 971

2. HK$14 391

Practise Now 14A

1. (a) 09 15 (b) 20 59
   (c) 00 10
2. (a) 12.08 a.m.
   (b) 2.10 a.m.
   (c) 12.56 p.m.
3. 00 00

Practise Now 14B

1. 1 hour 23 minutes
2. (i) 10 48
   (ii) 4 hours 3 minutes

Practise Now 15A

7 August 2023, 1.35 a.m.

Practise Now 15B

1. (a) $\frac{3}{10}$ h (b) 4.2 h
   (c) 0.2375 h
2. (a) 129 min (b) 4 min 24 s
   (c) 27 min (d) 1260 s

Practise Now 16

1. (i) 40.32 km/h
   (ii) 11.2 m/s
2. $11\ 458\frac{1}{3}$ m
3. (i) 33.3 m/s
   (ii) 200 000 cm/min
4. 14.2 times

Practise Now 17

16.0 km/h

Practise Now 18

5 hours

Exercise 9A

1. (i) 14 : 25 (ii) 25 : 39
2. (a) 1 : 6 (b) 7 : 3
   (c) 9 : 17 (d) 2 : 5
3. (a) 1 : 2 (b) 1 : 2
   (c) 2 : 5 (d) 3 : 5
4. (a) 96 (b) $1\frac{1}{2}$
5. $154
6. (i) 25 : 40 : 44
   (ii) 8 : 5 (iii) 10 : 11
7. (a) 16 : 36 : 15
   (b) 36 : 21 : 14
   (c) 11 : 21 : 60
   (d) 2 : 10 : 9
8. (a) 145 : 280 : 26
   (b) 8 : 2 : 3
   (c) 40 : 3000 : 81
   (d) 190 : 21 : 150
9. (i) 544 (ii) $953.70
10. (i) 112 : 3 (ii) 75 : 210 : 4
11. (a) 12 : 53 (b) 100 : 63
    (c) 1 : 5 (d) 100 : 1
12. (a) 15 : 2 (b) 750 : 11
    (c) 15 : 1 (d) 41 : 15
13. (a) 15 : 16 (b) 6 : 1
14. (i) 80 (ii) 10
15. (i) 6 : 16 : 9 (ii) 16 : 9
16. $x = 2\frac{1}{2}$, $y = 6\frac{3}{4}$
17. 450
18. (i) 4
    (ii) Lemonade: 630 ml, carbonated water: 840 ml
19. 7
20. $\frac{40}{89}$
21. (a) $x = 10, y = 8, z = 6$
    (b) $x = 0.5, y = 0.4, z = 0.3$

Exercise 9B

1. (a) 30 (b) $0.29
   (c) $1600
   (d) $4\frac{8}{13}$ kg or 4.62 kg
2. Kumar
4. $55.20
5. (i) 743.4 km
   (ii) $469.18
6. (i) 350 g (ii) 18 m^2
7. (i) PKR 225 822
   (ii) PKR 49 155
8. Brand C
9. (i) 269°
   (ii) $31\frac{5}{8}$ minutes
10. 15
11. (i) $1680 (ii) $2040
12. NZ$486.43
13. (i) RS Forex (ii) RS Forex
14. (i) Website G
    (ii) PKR 19 609

Exercise 9C

1. (a) 5 hours 14 minutes
   (b) 8 hours 42 minutes
   (c) 4 hours 57 minutes
   (d) 7 minutes
2. 17 35
3. 4 hours 15 minutes
4. (a) $\frac{49}{60}$ h (b) 0.6 h
   (c) 1.1 h (d) 2.9 min
   (e) 0.2625 h (f) $\frac{167}{400}$ h
5. (a) 7.8 s (b) 20 min
   (c) 6600 s (d) 3456 s
6. (i) 49.2 km/h
   (ii) $13\frac{2}{3}$ m/s or 13.7 m/s
7. 360 000 m
8. (a) 504 km/h
   (b) 1134 km/h
   (c) 14.52 km/h
   (d) 4.5 km/h
9. (a) 0.65 m/s
   (b) 102 m/s
   (c) $\frac{1}{6}$ m/s or 0.167 m/s
   (d) $1433\frac{1}{3}$ m/s
10. 10.5 times
11. 88.8 km/h
12. 7.19 p.m.
13. 21 00
14. 80 km/h
15. (i) 4 seconds
    (ii) $7\frac{1}{2}$ m/s
16. 16 m/s
17. 84.3 km/h
18. Possible values: $x = 50$, $y = 80$
19. 280 m
20. 37 800 m
21. 37.9 km/h

Chapter 10 Basic Geometry

Practise Now 1A

(a) Acute (b) Reflex
(c) Obtuse (d) Straight
(e) Right (f) Reflex
(g) Obtuse (h) Reflex
(i) Acute

Practise Now 1B

1. (a) 58 (b) 19.9°
2. 16

Practise Now 2

1. 22
2. 45

Practise Now 3

(a) 143°
(b) $a = 10, b = 25$

Practise Now 4A

(a) (i) $\angle a$ and $\angle m$, $\angle b$ and $\angle n$, $\angle c$ and $\angle o$, $\angle d$ and $\angle p$, $\angle e$ and $\angle i$, $\angle f$ and $\angle j$, $\angle g$ and $\angle k$, $\angle h$ and $\angle l$
(ii) $\angle c$ and $\angle m$, $\angle d$ and $\angle n$, $\angle g$ and $\angle i$, $\angle h$ and $\angle j$
(iii) $\angle c$ and $\angle n$, $\angle d$ and $\angle m$, $\angle g$ and $\angle j$, $\angle h$ and $\angle i$
(b) (i) No (ii) Yes
(iii) No

Practise Now 4B

1. $x = 74, y = 100, z = 80$
2. $h = 19.5, k = 39$

Practise Now 5

1. 282.2°
2. 108

Practise Now 6

65°

Exercise 10A

1. (a) $a = 79, b = 106, c = 98$
   (b) $d = 50, e = 227$
   (c) $f = 117, g = 45$
   (d) $h = 244, i = 94, j = 56$
2. (a) Obtuse (b) Reflex
   (c) Acute (d) Right
   (e) Straight (f) Reflex
   (g) Acute (h) Obtuse
3. (a) 72° (b) 44°
   (c) 37° (d) 26°
4. (a) 144° (b) 168°
   (c) 78° (d) 9°
5. (a) $a = 147$ (b) $b = 65$
   (c) $c = 20$ (d) $d = 25$

6. (a) 49 (b) 30
7. (a) $a = 106$ (b) $b = 30$
(c) $c = 11.25$ (d) $d = 11$
8. (i) 48° (ii) 42°
9. (a) $a = 47$
(b) $b = 18$, $c = 126$
10. (a) 90 (b) 60
11. $A\hat{O}B = 30°$, $B\hat{O}C = 60°$, $C\hat{O}D = 120°$, $D\hat{O}A = 150°$
12. (a) $a = 11$, $b = 45$
(b) $c = 23$, $d = 69$, $e = 111$
(c) $f = 22$, $g = 22$, $h = 44$
(d) $i = 6$, $j = 23$
13. (i) $x = 22$, $y = 16$
(ii) 132°; 278°

Exercise 10B

1. (a) (i) $B\hat{X}R$ and $D\hat{Z}R$, $A\hat{X}R$ and $C\hat{Z}R$, $A\hat{X}S$ and $C\hat{Z}S$, $B\hat{X}S$ and $D\hat{Z}S$, $B\hat{W}P$ and $D\hat{Y}P$, $A\hat{W}P$ and $C\hat{Y}P$, $A\hat{W}Q$ and $C\hat{Y}Q$, $B\hat{W}Q$ and $D\hat{Y}$
Q
(ii) $A\hat{X}S$ and $D\hat{Z}R$, $B\hat{X}S$ and $C\hat{Z}R$, $A\hat{W}Q$ and $D\hat{Y}P$, $B\hat{W}Q$ and $C\hat{Y}P$
(iii) $A\hat{X}S$ and $C\hat{Z}R$, $B\hat{X}S$ and $D\hat{Z}R$, $A\hat{W}Q$ and $C\hat{Y}P$, $B\hat{W}Q$ and $D\hat{Y}P$
(b) No (c) No
2. (a) $a = 117$, $b = 117$, $c = 63$, $d = 78$
(b) $e = 31$, $f = 66$
(c) $g = 83$, $h = 69$
(d) $i = 45$, $j = 60$
3. (a) $a = 38$, $b = 34$
(b) $c = 20$, $d = 70$
(c) $e = 18$ (d) $f = 29$
4. (i) 68° (ii) 54°
5. (i) 86° (ii) 141°
6. (a) $a = 106$ (b) $b = 104$
7. $x = 65$, $y = 40$
8. (i) 52° (ii) 56°
(iii) 262°
9. $x = 147$, $y = 59$
10. 277
11. $a = 121$, $b = 41$
12. 23°
13. 46°
14. $w + x = y + z$

Chapter 11 Polygons and Geometrical Constructions

Practise Now 1

1. 42
2. 64°

Practise Now 2

1. (a) 109 (b) 95°
2. (a) 134° (b) 54

Practise Now 3

1. 75°
2. 110°

Practise Now 4

1. (i) 54° (ii) 27°
2. (i) 39° (ii) 73°

Practise Now 5

1. (i) 116° (ii) 84°
2. (i) 12 (ii) 34°

Practise Now 6

1. (i) 25° (ii) 72°
2. (ii) 2 cm

Practise Now 7

$YZ = 6.1$ cm

Practise Now 8

$\angle QPR = 77°$

Practise Now 9

1. 3.4 cm and 9.3 cm
2. 11.3 cm

Practise Now 10

1. 12.2 cm
2. 12.3 cm

Practise Now 11

1. (i) 7.0 cm (ii) 54°
2. 79°

Practise Now 12

1. (a) $a = 104$ (b) $b = 36$
2. 5

Practise Now 13

(i) 3960° (ii) 165°

Practise Now 14

1. (a) 9 (b) 180
2. 144°
3. 17

Practise Now 15

60°

Practise Now 16

(i) 108° (ii) 36°
(iii) 162° (iv) 9°
(v) 20

Exercise 11A

1. (a) $a = 74$ (b) $b = 66$
(c) $c = 22.5$ (d) $d = 26.5$
(e) $x = 60$ (f) $y = 25.7$
2. $a = 75$, $b = 23$
3. (i) 114 (ii) 79°
4. $a = 25$, $b = 97$
5. (i) 35° (ii) 125°
6. 62°
7. 81°
8. 138
9. 53°
10. $S\hat{U}T = 79°$, $R\hat{S}U = 44°$
11. $x = 39$, $y = 77$
12. 15°
13. 72°

Exercise 11B

1. (a) $a = 36$, $b = 36$
(b) $c = 51$, $d = 51$
2. (i) 64° (ii) 26°
3. (i) 62° (ii) 75°
4. (a) $p = 38$, $q = 104$
(b) $r = 42$, $s = 48$
5. (a) $a = 106$, $b = 48$
(b) $c = 20$, $d = 40$
6. (a) $a = 40$, $b = 58$
(b) $c = 114$
7. (i) 65° (ii) 46°
8. $x = 30$, $y = 114$
9. (a) $a = 37$, $b = 127$
(b) $c = 67.5$, $d = 22.5$
10. (i) 88° (ii) 23°
11. (i) 115° (i[illegible]
12. (i) 54° (ii) 72°
(iii) 36°
13. (i) 110° (ii) 28°
14. 6
15. (i) 58° (ii) 96°
16. 40°
17. (i) 31° (ii) 97°
18. (i) 59° (ii) 31°
20. 360°

Exercise 11C

1. 9.1 cm
2. 53°
3. (a) 9.4 cm (b) 6.0 cm
4. 16.9 cm
5. 3.9 cm, 6.9 cm
6. (i) 5.4 cm (ii) 68°
7. (i) 4.2 cm (ii) 20°
8. $\angle CED$, 53°
9. 1.5 cm and 9.3 cm
10. (ii) 171°
11. 133°
12. (i) 104° (ii) 4.0 cm
15. (ii) 13.6 cm, 119°; 10.2 cm, 139°

Exercise 11D

1. (a) 1620° (b) 1800°
(c) 2340° (d) 3240°
2. (a) $a = 110$ (b) $b = 66$
(c) $c = 83$ (d) $d = 30$
3. (a) (i) 1080°
(ii) 135°
(b) (i) 2880°
(ii) 160°
4. (a) 8 (b) 4
(c) 90 (d) 3
5. (a) 9 (b) 20
(c) 45 (d) 72
6. (a) 165° (b) 170°
7. 8
8. 21
9. 15
10. 77.1°
11. (i) 120° (ii) 60°
(iii) 60° (iv) 108°
(v) 72° (vi) 24°
12. (i) 10 (ii) 126°
(iii) 144°

13. (i) 162° (ii) 153°
15. 180°
16. (i) 12 (ii) 135°
(iii) 120°
17. $x = 10$, perimeter = 108 m;
$x = 9$, perimeter = 120 m

Chapter 12 Perimeter and Area of Plane Figures

Practise Now 1
(a) 100 000 cm^2
(b) 225 000 cm^2
(c) 1600 cm^2 (d) 0.03 m^2
(e) 0.7146 m^2 (f) 0.000 01 m^2

Practise Now 2
203 m^2

Practise Now 3B
(a) 3 cm^2 (b) 6.75 cm^2
(c) 4.8 m^2

Practise Now 4
1. 16.8 m
2. (i) 252 cm^2 (ii) 16.8 cm
3. 209.5 m^2

Practise Now 5B
1. (i) 74 m (ii) 168 m^2
2. (i) 32 cm (ii) 59.2 cm^2

Practise Now 6
9.6 m

Practise Now 7
5 : 1

Practise Now 8
12 : 2 : 1

Practise Now 9B
(i) 29.7 m (ii) 36.4 m^2

Practise Now 10
(i) 6 m (ii) 7.2 m

Practise Now 11A
(i) *AD*, *CF*
(ii) *OA*, *OD*, *OC*, *OF*
(iii)(a) =
(b) =
(c) <

Practise Now 11B
1. 37.7 cm; 113 cm^2
2. 101 cm^2; 41.1 cm

Practise Now 12
1. 101 cm
2. (i) 5.97 cm
(ii) 37.5 cm
(iii) 112 cm^2

Practise Now 13
(i) 94.0 cm (ii) 462 cm^2
(iii) 322 cm^2

Exercise 12A
1. (a) 400 000 cm^2
(b) 892 000 cm^2
(c) 300 cm^2
(d) 51 760 cm^2
2. (i) 14 cm (ii) 65 cm
3. (a) 3.15 cm^2 (b) 13.5 cm^2
4. 1.86 cm
5. 350 m^2
6. 157.5 m^2
7. 17.6 cm
8. 235.75 m^2
9. (a) 6 cm by 5 cm, 10 cm by 3 cm, 15 cm by 2 cm
(b) 12 cm by 8 cm, 15 cm by 5 cm, 17 cm by 3 cm

Exercise 12B
1. (i) 32 cm (ii) 54 cm^2
2. (i) 22.4 cm (ii) 28 cm^2
3. (a) 84 cm^2 (b) 7 m
(c) 5.5 mm
4. 14 m
5. 28 cm
6. (i) 89.5 cm (ii) 416.25 cm^2
7. (a) 54 cm^2 (b) 14 m
(c) 13 mm
8. (i) 25 m (ii) 11 m
9. 351 cm^2
10. 270 cm^2
11. (i) 80 cm^2 (ii) 20 cm^2
12. 175 cm^2

Exercise 12C
1. (a) 50.3 cm; 201 cm^2
(b) 5.51 cm; 95.3 cm^2
(c) 9.80 mm; 61.6 mm
2. (i) 14.5 cm^2 (ii) 15.4 cm
3. (i) 9.99 cm (ii) 78.5 cm^2
4. (a) 14.3 m; 12.3 m^2
(b) 56.5 cm; 191 cm^2
5. (i) 28.6 m (ii) 50.4 m^2
6. 320 cm^2
7. 184 cm^2
8. (i) 863 cm^2
(ii) 1.85 % decrease
9. 10.8%
10. 0.250 cm^2
11. 99.0 minutes
12. 44.0 cm; 42.1 cm^2

Chapter 13 Statistical Data Handling

Practise Now 1A
(i) (a) \$5 500 000
(b) \$7 000 000
(ii) 2020; \$1 500 000

Practise Now 1B
1. (i) Black (ii) No
2. (ii) (a) 384
(b) 594
(iii) 39.3%
(iv) (a) 17.0%
(b) No
(v) December

Practise Now 1D
(i) 20.4
(ii) $22\frac{2}{3}$% or 22.7%
(iii) 1150 ml

Exercise 13A
1. (i) June and July; 250
(ii) 900 (iii) \$315 000
(iv) 30%
2. (ii) 4 : 5
(iii) $83\frac{1}{3}$% or 83.3%
4. (i) 95 (ii) 10.5%
5. (i) 53 (ii) 50%
6. (i) 13, 16
(ii) (a) 68 (b) 64
(iii) 17.6% (iv) 40%
(v) No
7. (i) 950 (ii) 500
(iii) 20.4%

Exercise 13B
2. (i) 90° (ii) 100°
(iii) $27\frac{7}{9}$% or 27.8%
(iv) 36
3. (i) 50% (ii) 20%
(iii) 63
4. (i) 120 (ii) 375
(iii) $45\frac{5}{6}$% or 45.8%
(iv) 60°, 75°, 60°, 90°, 75°
5. (i) 20°
(ii) (a) 225 (b) 540
(iii) 31.5%
6. 9
7. No